More properties of spectra; homotopy limits + colimits

$$\Sigma^\infty : Top_* \rightleftarrows Sp : Ev_0$$

$$\mathbb{L}\Sigma^\infty : Ho(Top_*) \rightleftarrows Ho(Sp) \quad \mathbb{R}Ev_0 = \Omega^\infty.$$

Recall: a spectrum is fibrant in the stable model structure if it's an Ω-spectrum

$$X_n \xrightarrow{\sim} \Omega X_{n+1}$$

Constructing fibrant replacements:

Given X, define

$$(R_k X)_n = \Omega^k X_{n+k}.$$

$$\begin{tikzcd}
(R_k X)_n \arrow[r] \arrow[d, equal] & \Omega (R_k X)_{n+1} \arrow[d, equal] \\
\Omega^k X_{n+k} \arrow[dr] & \Omega \Omega^k X_{n+k+1} \arrow[d, equal] \\
& \Omega^{k+1} X_{n+k+1}
\end{tikzcd}$$

$$\begin{aligned}
\pi_j R_k X &= \operatorname{colim} \pi_{n+j} (R_k X)_n \\
&= \operatorname{colim} \pi_{n+j} \Omega^k X_{n+k} \\
&= \operatorname{colim} \pi_{n+j+k} X_{n+k} \\
&= \pi_j X
\end{aligned}$$

$$X \xrightarrow{\sim} R_k X \xrightarrow{\sim} R_{k+1} X \longrightarrow \cdots$$

$$(R_\infty X)_n = \operatorname{hocolim}_k (R_k X)_n$$

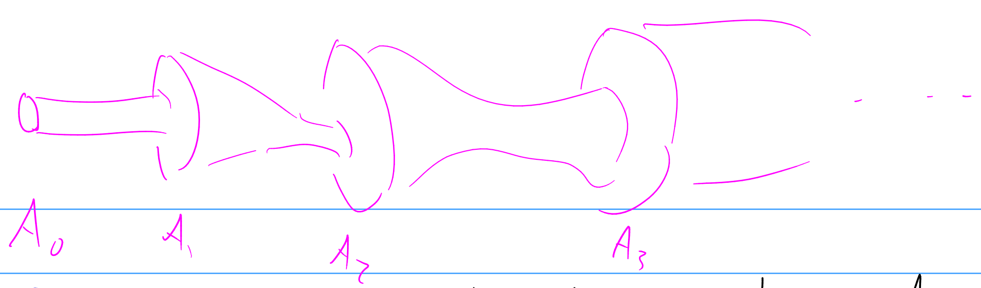
We can define this using the mapping telescope:

Given pointed cofibrant spaces A_n indexed by $n \in \mathbb{N}$,

$$\operatorname{hocolim} A_n = \coprod A_n \times [0,1] \Big/ (x_n, 1) \sim (f_n(x_n), 0)$$

$(f_n : A_n \to A_{n+1})$.

$* \times [0,1] \sim *$.



Properties. $\pi_* \operatorname{hocolim} A_n = \operatorname{colim} \pi_* A_n$

$\Omega \operatorname{hocolim} A_n \simeq \operatorname{hocolim} \Omega A_n$

$X \xrightarrow{\sim} R_\infty X$ induces an iso on π_*.

$$(R_\infty X)_n = \operatorname{hocolim} \underline{\Omega^k X_{n+k}} \xrightarrow{\sim} \operatorname{hocolim} \underline{\Omega^{k+1} X_{n+k+1}}$$
$$\xrightarrow{\sim} \Omega \operatorname{hocolim} \Omega^k X_{n+k+1}$$
$$= \Omega (R_\infty X)_{n+1}$$

$$(R_\infty X)_0 = \operatorname{hocolim}_k \Omega^k X_k.$$

ex. $\Omega^\infty HA = K(A, 0)$.

$\Omega^\infty KU = \mathbb{Z} \times BU$.

$\Omega^\infty \Sigma^\infty X = \operatorname{hocolim} \Omega^n \Sigma^n X$. (for $X \in Top_*$).

A space X is an infinite loop space if there are spaces $B^n X$ for every $n \in \mathbb{N}$ and equivalences

$$\underline{X \xrightarrow{\sim} \Omega^n B^n X}.$$

If X is an infinite loop space, choose deloopings $B^n X$ so that $\pi_* B^n X = 0$ in $* = 0, \ldots, n-1$.

$\pi_* B^n X = \pi_{*-n} X$ for $* \geq n$.

$$\underline{B^n X \simeq \Omega B^{n+1} X}.$$

There's a Ω-spectrum $B^\infty X$ with

$$(B^\infty X)_n = B^n X.$$

There is a Quillen equivalence

$$B^\infty : (\Omega^\infty\text{-Spaces}) \rightleftarrows Sp_{\geq 0} : \Omega^\infty.$$

Def. A spectrum X is <u>0-connective</u> if $\pi_* X = 0$ in degrees $* < 0$.

$Sp_{\geq 0}$ = category of 0-connective spectra.

Homotopy limits and colimits

Limits + colimits exist in Sp:

$$\Big(\lim_{\alpha \in I} X_\alpha\Big)_n = \lim_{\alpha \in I} (X_\alpha)_n$$

$$\begin{tikzcd}
(\lim X_\alpha)_n \arrow[r] \arrow[ddr] & \Omega(\lim X_\alpha)_{n+1} \arrow[d, phantom, "\cong" rotate=90] \\
& \Omega \lim (X_\alpha)_{n+1} \arrow[d, phantom, "\cong" rotate=90] \\
& \lim \Omega(X_\alpha)_{n+1}
\end{tikzcd}$$

In Top_*:

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & * \arrow[d] \\
* \arrow[r] & * \arrow[ul, phantom, "\ulcorner"]
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[r] \arrow[d] & CX \arrow[d] \\
CX \arrow[r] & \Sigma X . \arrow[ul, phantom, "\ulcorner"]
\end{tikzcd}$$

Idea. Let $\{X_\alpha\}_{\alpha \in I}$. Then $\operatorname{hocolim}_{\alpha \in I} X_\alpha$ is the universal object equipped with a <u>homotopy coherent</u> cocone from the diagram $\{X_\alpha\}$.

- $X_\alpha \xrightarrow{f_\alpha} X$.
- $$\begin{tikzcd}
X_\alpha \arrow[d, "g_{\alpha\beta}"'] \arrow[dr, "f_\alpha"] & \\
X_\beta \arrow[r, "f_\beta"'] \arrow[ur, phantom, "\Nearrow"] & X
\end{tikzcd}
\qquad f_\beta \circ g_{\alpha\beta} \sim f_\alpha .$$
- $$\begin{tikzcd}
X_\alpha \arrow[rr, "f_\alpha"] \arrow[dr] \arrow[dd] & & X \\
& X_\beta \arrow[ur, "f_\beta"'] \arrow[dl] & \\
X_\gamma \arrow[uurr, "f_\gamma"'] & &
\end{tikzcd}$$
  There's a 2-cell connecting all the homotopies.
- etc.

Remarks: $\operatorname{hocolim} X_\alpha$ only defined up to homotopy.

Not the same as the colimit in the homotopy category.

This makes sense in any model category.

$\mathcal{M}$ = model category $\qquad I$ = diagram category

If $\mathcal{M}$ is cofibrantly generated, there's a Quillen adjunction

$$\operatorname{colim} : \mathcal{M}^I \rightleftarrows \mathcal{M} : \text{const}.$$

$\mathcal{M}^I$ has the projective model structure, in which fibrations + WEs are levelwise.

$$\operatorname{hocolim} = \mathbb{L}\operatorname{colim} : Ho(\mathcal{M}^I) \longrightarrow Ho(\mathcal{M}).$$

Reference: Dugger, A primer on homotopy colimits.

ex. • I discrete: $\{X_\alpha\}$ is cofibrant if it is objectwise.

• $I = \mathbb{N}$: $\{X_0 \to X_1 \to \cdots\}$ is cofibrant if each X_i is cofibrant, and all maps are cofibrations.

• $I = \bullet \leftarrow \bullet \rightarrow \bullet$: a diagram is cofibrant if all objects are cofibrant and all maps are cofibrations

(Possible to weaken this to: all objects cofibrant, one map is a cofibration.)

Given
$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \\ * & \end{tikzcd}$$
in Top_*,

Replace X and Y by CW-complexes, f by a CW-inclusion, $*$ by CX.

$$\begin{tikzcd} X \arrow[r, hook, "f"] \arrow[d, hook] & Y \arrow[d] \\ CX \arrow[r] & Y \coprod_X CX = C(f). \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

$$\begin{tikzcd} X \arrow[r] \arrow[d] & * \arrow[d] \\ * \arrow[r] & \Sigma X \arrow[ul, phantom, "\ulcorner h", near start] \end{tikzcd} \rightsquigarrow \begin{tikzcd} X \arrow[r, hook] \arrow[d, hook] & CX \arrow[d] \\ CX \arrow[r] & \Sigma X. \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

In general, define
$$\begin{tikzcd} X \arrow[rr] \arrow[dr, hook, bend right] & & * \\ & CX \arrow[ur, "\sim"'] & \end{tikzcd}.$$

This works in any model category.

We'll say that
$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] \\ Z \arrow[r] & W \end{tikzcd}$$
is a <u>homotopy pushout square</u>

if the natural map
$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] \arrow[ddr, bend left] & \\ Z \arrow[r] \arrow[drr, bend right] & \text{hocolim} \arrow[dr, "\sim"] & \\ & & W \end{tikzcd}.$$

$$\begin{tikzcd} \Omega X \arrow[r] \arrow[d] & * \arrow[d] \\ * \arrow[r] & X \end{tikzcd}$$

<u>Exercise:</u> show that Σ and Ω, defined this way, are equivalent to the shift functors on $Ch(R)$.

<u>Thm</u>. In Sp, a square
$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] \\ Z \arrow[r] & W \end{tikzcd}$$
is a homotopy pushout square iff it's a homotopy pullback square.

$$\begin{tikzcd} \Omega\Sigma X \simeq X \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner" very near start, "\ulcorner" very near end] & * \arrow[d] \\ * \arrow[r] & \Sigma X \end{tikzcd}$$

$$\mathcal{C} \rightsquigarrow \{ X_n, \quad X_n \xrightarrow{\simeq} \Omega X_{n+1} \},$$
$$= Sp(\mathcal{C}).$$

$$[\Sigma^\infty K, \quad E]$$

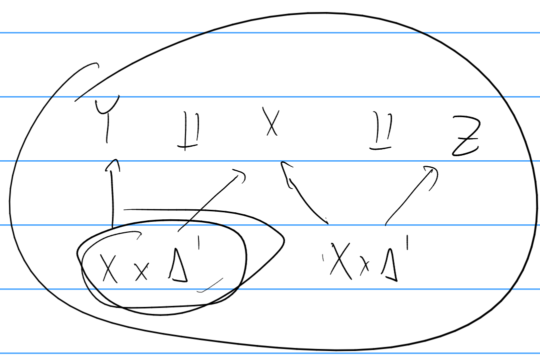
$$[\Sigma^\infty K, \quad \operatorname{hocolim} E_\alpha] = [K, \quad \Omega^\infty \operatorname{hocolim} E_\alpha].$$

$$E^*(\operatorname{hocolim} K_\alpha) = \text{derived limit of } E^*(K_\alpha)$$

$$(\operatorname{holim} E_\alpha)^*(K) = \text{derived limit of } E_\alpha^* K.$$

$$\begin{tikzcd} X \arrow[r] \arrow[d] \arrow[dr] \arrow[rd, bend left] \arrow[dr, bend right] & Y \arrow[d] \arrow[dl, Rightarrow] \\ Z \arrow[r] \arrow[ur, Rightarrow] & W \arrow[d] \\ & W' \end{tikzcd} \qquad \begin{tikzcd} S^1 \arrow[r] \arrow[d] & * \arrow[d] \\ * \arrow[r] & * \end{tikzcd} \qquad \begin{tikzcd} S^1 \arrow[r] \arrow[d] & * \arrow[d] \\ * \arrow[r] & W \end{tikzcd}$$

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "g"'] & Y \\ Z \end{tikzcd}$$



$$Y \coprod_X X \times \Delta^1 \coprod_X Z$$

$$\begin{tikzcd} X \amalg X \arrow[r, "\nabla"] \arrow[d, hook] & X \\ Cyl(X) \arrow[ur, "\sim"'] \end{tikzcd}$$

Presheaves of Spaces on $\underline{\underline{Sm\,Sch/_k}}$

$\downarrow$

Sheaves

$\downarrow$

$L_{\mathbb{A}^1}$ $\qquad X \simeq \mathbb{A}^1 \times X.$

constant sheaf S^p $\qquad \pi_{p,q} X = [S^p \wedge (\mathbb{P}^1)^{\wedge q}, X].$

$\mathbb{P}^1$

$$\pi_n X = \operatorname{colim} \pi_{n+k} X_k$$

$$\pi_n X = [\Sigma^\infty S^n, X].$$