Parametrizations of Lagrangian Submanifolds

Let $\mathbb{R}^n \times \mathbb{R}^n = T^*\mathbb{R}^n$, equipped with the form

$$\sigma\big((x,\xi) ; (y,\eta) \big) = \langle \xi, y \rangle - \langle \eta, x \rangle$$

If A is a $n \times n$ matrix and

$$\Lambda = \{ (x, Ax) \}$$

then Λ is a Lagrangian subspace $\Leftrightarrow$ A is symmetric

Indeed,

$$\sigma\big((x, Ax) ; (y, Ay) \big) = \langle Ax, y \rangle - \langle Ay, x \rangle = 0$$

$$\Leftrightarrow A^t = A.$$

On the other hand, if $\Lambda \subset T^*\mathbb{R}^n$ has dimension N, then Λ is given by a system of Equations

$$Mx + N\xi = 0 \qquad M, N \quad n \times n \text{ matrices}$$

But then $N\xi = -Mx$

So $\Lambda = \{ (x, Ax) \} \Leftrightarrow$

$$\Lambda \cap \{ (0, \xi) \} = \{ 0 \}$$

We say that Λ and $\{ (0,\xi) \}$ intersect transversally

Let $\Lambda_A = \{ (x, Ax) \}$ and let

Λ be a n-dimensional subspace of $T^* \mathbb{R}^n$.

$\Lambda = \{ Mx + N\xi = 0 \}$

The intersection of Λ_A and Λ is not

transversal if

$$M x + N A x = 0$$

has more than the <u>trivial solution</u>. $x = 0$

$$\det (M + N A) = 0$$

<u>Proposition</u>: Let $\Lambda_0 \subset T^* \mathbb{R}^n$ be a Lagrangian subspace. There exists an invertible $n \times n$ matrix H such that after the canonical change of variables

$$(x, \xi) \longmapsto (Hx, {}^t H^{-1} \xi)$$

Λ_0 takes the form

$$x' = 0 ; \quad \xi'' = B x'' \qquad B\text{ - symmetric.}$$

$$x = (x', x'') \; ; \; \xi = (\xi', \xi'')$$

$$x' = (x_1, \dots, x_{n-d}) \; ; \; x'' = (x_{n-d+1}, \dots, x_n).$$

$$0 \le d \le n$$

Notice that any change of variables,

$x \longmapsto Hx$, induces a symplectic

change $(x, \xi) \longmapsto (Hx, {}^tH^{-1}\xi)$

Since

$$\sigma\left((Hx, {}^tH^{-1}\xi) ; (Hy, {}^tH^{-1}\eta) \right) =$$

$$= \langle {}^tH^{-1}\xi , Hy \rangle - \langle {}^tH^{-1}\eta ; Hx \rangle$$

$$= \langle \xi, y \rangle - \langle \eta, x \rangle$$

Proof Let $L \subset \mathbb{R}^n_x$ be the projection of Λ_0 onto $\mathbb{R}^n_x$. If $L = \{0\}$, we are done. $d = 0$ and $\xi = Bx$.

If L has dimension d, make a linear change of variables in x such that

$$L = \{ x' = 0 \}$$

Suppose this transformation is $x \longmapsto Mx$, then make a change $\xi \longmapsto {}^tM^{-1}\xi$.

We claim that in these coordinates the map

$$\pi : \Lambda_0 \longrightarrow \mathbb{R}^n$$
$$(x', x'', \xi', \xi'') \longmapsto (x'', \xi')$$

is an isomorphism. In other words, $\operatorname{Ker} \pi = 0$.

Suppose $(x_0', x_0'', \xi_0', \xi_0'') \in \operatorname{Ker} \pi$, then

$$(x_0', 0, 0, \xi_0'') \in \Lambda.$$

By definition of x', $x_0' = 0$. So $(0, 0, 0, 0, \xi_0'') \in \Lambda$

But then $\sigma\left((0,0,0,\xi_0'') ; (x', x'', \xi', \xi'') \right) = \langle \xi_0'', x'' \rangle = 0$

for all x'' $\Rightarrow$ $\xi_0'' = 0$.

Conclusion: Λ_0 is parametrized by (x'', ξ')

and so

$$\Lambda_0 = \{ x' = 0 ; \ \xi'' = Bx'' + C\xi' \}$$

In this case $(0, 0, \xi_0' ; C\xi_0') \in \Lambda$ and

$$0 = \sigma\left((0,0,\xi_0', C\xi_0') ; (0, x'', \xi', C\xi') \right) = \langle C\xi_0', x'' \rangle$$
$$\forall x''$$

$$C\xi_0' = 0 \quad \forall \xi_0' \Rightarrow C = 0$$

B is symmetric since Λ_0 is Lagrangian.

Suppose we are working in coordinates where

$$\Lambda_0 = \{ (0, x'', \xi'; Bx'') \} = \{ x' = 0, \ \xi'' = Bx'' \}$$

Let $\Lambda_A = \{ x' = 0, \ \xi'' = D x'' \}$

then Λ_A and Λ_0 are transversal $\Leftarrow\!\!=$

$$\det (B - D) \neq 0$$

Hence, given Λ_0, there exist infinitely many Lagrangians transversal to Λ_0.

<u>Theorem</u>: Let $\Lambda \subset T^*X$ be a Lagrangian submanifold and let $(x_0, \xi_0) \in \Lambda$, $\xi_0 \neq 0$. There exists a change of variables $y = \psi(x)$ in a neighborhood of x_0 such that if (y, η) are the corresponding canonical coordinates, then near $y = 0$, $\eta = (0, \dots, 1)$

$$\Lambda = \left\{ y = -\frac{\partial H}{\partial \eta} \right\}$$

for some C^∞ function $H(\eta)$.

Proof:

Step 1 : Choose x-coordinates near x_0 such that $\xi_0 = (0, 0, \dots, 1)$, $x_0 = 0$.

$$\Lambda_0 = T_{(x_0,\xi_0)} \Lambda \quad ; \quad T_{(x_0,\xi_0)}(T^* X) \equiv T^* \mathbb{R}^n$$

Then by the proposition, there exists a symmetric matrix A such that

$$\Lambda_A = \{ \xi = A x \} \quad \text{is transversal to } \Lambda_0$$

Claim: Λ_A is the tangent space to the Lagrangian.

$$\Lambda_\varphi = \{ \xi = d\varphi \} \quad \text{where}$$

$$\varphi = x_n + \frac{1}{2} \langle Ax, x \rangle$$

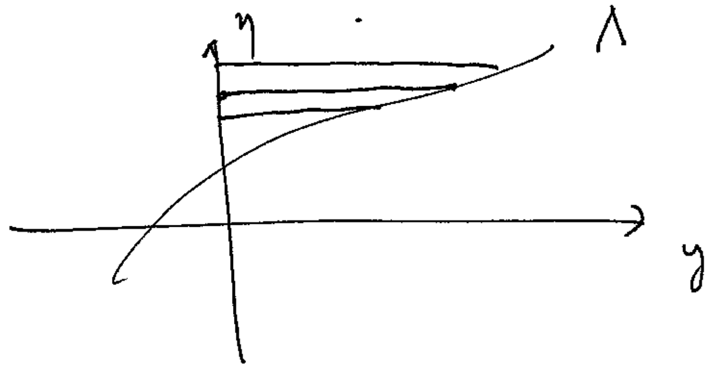
Therefore Λ and Λ_φ intersect transversally at (x_0, ξ_0). Now choose coordinates

$$y_j = x_j \quad 1 \le j \le n-1, \quad y_n = x_n + \frac{1}{2}\langle Ax, x\rangle$$

Let $(y_1, \dots, y_n, \eta_1, \dots, \eta_n)$ be the corresponding canonical coordinates $y = \Psi(x)$; $\eta = ({}^t\Psi'(x))^{-1} \xi$.

Therefore Λ is transversal to

$$\Lambda_\varphi = \{ \eta_n = 1 \} \text{ at } (0, \bar{\eta}), \quad \bar{\eta} = (0, 0, \dots, 1)$$



Hence
$$\pi : \Lambda \longrightarrow \mathbb{R}^n$$
$$(y, \eta) \longrightarrow \eta$$

is a diffeomorphism. This implies that

$$\Lambda = \{ y = \psi(\eta) \}.$$

<u>Exercise:</u> Show that $\Lambda = \{ y = -\frac{\partial}{\partial \eta} H(\eta) \}$.