Lecture 23 The principal symbol

1. Symbols on conic manifolds: Let V be a conic manifold and let

$$M_t : V \longrightarrow V$$

be the associated multiplication by real numbers $t>0$. We say that $a \in S^m(V)$ if the functions

$$F(t, x) = t^{-m} M_t^* a$$

are uniformly bounded in $C^\infty(V)$ for $t \geq 1$.

Example: $V = B(x_0, \varepsilon) \times \Gamma_\varepsilon$.

$$F(t, x, \xi) = t^{-m} a(x, t\xi)$$

$$\partial_t F = -m t^{-m-1} a(x, t\xi) + t^{-m} (\xi \partial_\xi a)(x, t\xi).$$

$$|\partial_t F(t, x, \xi)| \leq C$$

In general $|\partial_t^k F(t, x, \xi)| \leq C_k$.

Lemma: Let $\Gamma_j \subset \mathbb{R}^{n_j} \times (\mathbb{R}^{N_j} \setminus 0)$, $j = 1, 2$, be conic sets, and let $\psi : \Gamma_1 \to \Gamma_2$ be a proper C^∞ map commuting with multiplication by positive scalars in the second variable.

If $a \in S^m(\mathbb{R}^{n_2} \times \mathbb{R}^{N_2})$ has support in the interior of a compactly based cone $\mathcal{O}_2 \subset \Gamma_2$ then $a \circ \psi \in S^m(\mathbb{R}^{n_1} \times \mathbb{R}^{N_1})$ if the composition is defined as 0 outside Γ_1.

Proof: The composition $a \circ \psi$ is supported in a compactly based cone $\mathcal{O}_1 \subset \Gamma_1$ where $\psi(x, \xi) = (y, \eta)$ implies $|\xi|/C \le |\eta| \le C|\xi|$. The hypothesis on a means that

$$|D^\alpha_{y,\eta}\, a(y, t\eta)| \le C_\alpha t^m \qquad 1/C \le |\eta| \le C.$$

Since $a \circ \psi(x, t\xi) = a(x, t\,\psi(x,\xi))$, we obtain

$$|D^\alpha_{x\xi}(a \circ \psi)(x, t\xi)| \le C_\alpha t^m \qquad |\xi| = 1$$

If $V = \mathbb{R}^n \times (\mathbb{R}^N, 0)$ with variables x, θ

$|dx|^{1/2}\,|d\theta|^{1/2}$ is a $1/2$-density which is

homogeneous of degree $\mu = N/2$.

If $a \in S^m(V, \Omega_V^{1/2})$ $\quad a = a_0\, b.$

$b \in S^{m-\mu}(V)$ is a scalar symbol.

a_0 = fixed $1/2$-density.

Let $u \in I^m(X, \Lambda)$. For any $(x_0, \xi_0) \in \Lambda$

Choose local coordinates in X near x_0 such

that $\Lambda = \{x = H'(\xi)\}$ in a conic nbhd Γ of (x_0, ξ_0).

If $\Gamma_1 \subset\subset \Gamma$ is a compactly based cone.

one can split. $u = u_1 + u_2$.

$u_j \in I^m(X, \Lambda)$ $\quad WF(u_1) \subset \Gamma_1$

$WF(u_2) \cap \Gamma_1 = \phi$

with u_1 compactly supported

If $u = \tilde{u}_1 + \tilde{u}_2$ is another decomposition with the same properties we have

$$WF(\tilde{u}_1 - u_1) \cap \Gamma_1 = \phi \quad , \tilde{\Gamma}_1 = \Gamma_1 \cap \Lambda = \{ (H'(\xi), \xi) ; \xi \in \gamma_1 \}$$

The Fourier transform of $\tilde{u}_1 - u_1$ is rapidly decreasing in a conic neighborhood of γ_1.

Hence the

$$e^{iH(\xi)} \hat{u}_1(\xi) = (2\pi)^{n/4} v(\xi) \in S^{m-n/4}$$

and the class of v in $S^{m-n/4}(\gamma_1)/S^{-\infty}(\gamma_1)$ does not depend on the decomposition of u.

We shall consider

$$v(\xi) \, |d\xi|^{1/2} \in S^{m+n/4}(\tilde{\Gamma}_1, \Omega^{1/2}_{\Gamma_1}) / S^{-\infty}(\tilde{\Gamma}_1, \Omega^{1/2}_{\Gamma_1})$$

In view of the diffeomorphism

$$\gamma_1 \ni \xi \mapsto (H'(\xi), \xi) \in \tilde{\Gamma}_1 .$$

According to (*) if $u \in I^m_{comp}(X, \Lambda)$

and $e^{iH(\xi)} \hat{u}(\xi) = (2\pi)^{n/4} v(\xi)$, then

$$v(\xi) \, |d\xi|^{1/2} - a(x, \theta) \, e^{i \frac{\pi}{4} \operatorname{sgn} \Phi} \, |\det \Phi|^{-1/2} \, |d\xi|^{1/2}$$

$$\in S^{m + n/4 - 1} (\Lambda, \Omega_\Lambda^{1/2}).$$

Here $\varphi'_\theta(x, \theta) = 0$ and $\varphi'_x(x, \theta) = \xi$ defines

(x, θ) as a function of ξ.

Given a codimension k surface $\Sigma \subset \mathbb{R}^d$

$$\Sigma = \{ f_1 = \dots = f_k = 0 \}$$

there exists an $d-k$ form on Σ d_Σ.

such that

$$d_\Sigma \wedge df_1 \wedge \dots \wedge df_k = dx_1 \wedge \dots \wedge dx_d$$

If $(\lambda_1, \dots, \lambda_{d-k})$ are coordinates on Σ

$$d_\Sigma = F \, d\lambda_1 \wedge \dots \wedge d\lambda_{d-k}$$

where
$$F = \left| \frac{\partial(\lambda, f)}{\partial x} \right|^{-1}.$$

If we apply this to

$$C_\varphi = \{ (x,\theta) : \varphi'_\theta(x,\theta) = 0 \}$$

$$d_C = |d\lambda| \cdot \left| \frac{\partial(\lambda, \varphi'_\theta)}{\partial(x,\theta)} \right|^{-1}.$$

In particular if we take $\lambda = \varphi'_x(x,\theta) = \xi$.

$$d_C = |d\xi| . |\det \Phi|^{-1}.$$

and therefore.

$$b(\xi)\, |d\xi|^{1/2} \equiv a(x,\theta)\, d_C^{1/2}\, e^{i\frac{\pi}{4}\operatorname{sgn}\Phi}$$
$$\mod S^{m+n/4-1}(\Lambda, \Omega^{1/2}_\Lambda).$$

where $C = C_\varphi$ is identified with Λ by the map $(x,\theta) \mapsto (x, \varphi'_x)$.

If we now introduce new coordinates $\tilde{x}$ and transform $u(x)$ as a $1/2$-density:

$$\tilde{u}(\tilde{x}) = u(x(\tilde{x}))\left|\frac{Dx}{D\tilde{x}}\right|^{1/2}, \text{ then.}$$

$$\tilde{u}(\tilde{x}) = (2\pi)^{-\frac{(n+2N)}{4}} \int e^{i\tilde{\varphi}(\tilde{x},\theta)} \tilde{a}(\tilde{x},\theta)\, d\theta.$$

$$\tilde{\varphi}(\tilde{x},\theta) = \varphi(x(\tilde{x}),\theta) \qquad \tilde{a}(\tilde{x},\theta) = a(x(\tilde{x}),\theta)\left|\frac{Dx}{D\tilde{x}}\right|^{1/2}.$$

If we recall the definition of d_C,

$$d_C = \left|\frac{Dx}{D\tilde{x}}\right| d_{\tilde{C}}$$

and so $\tilde{a}\, d_{\tilde{C}}^{1/2} = a\, d_C^{1/2}$.

If $u \in I^m(X, \Lambda; \Omega_X^{1/2})$, then microlocally

$$v(\xi)\,|d\xi|^{1/2} \equiv a(x,\theta)\, e^{i\frac{\pi}{4}\operatorname{Sgn}\Phi}\, d_C^{1/2}$$

$$\equiv \tilde{a}(\tilde{x},\theta)\, e^{i\frac{\pi}{4}\operatorname{Sgn}\tilde{\Phi}}\, d_{\tilde{C}}^{1/2} \quad \operatorname{mod} S^{m+n/4-1}$$

Hence, after a change of variables.

$$x = x(\tilde{x})$$

$$\tilde{a}\, d_c^{1/2} \equiv e^{i\frac{\pi}{4}(\mathrm{Sgn}\,\Phi - \mathrm{Sgn}\,\tilde{\Phi})}\, a\, d_c^{1/2}.$$

$$\bmod S^{m+n/4-1}.$$

Φ and $\tilde{\Phi}$ are ~~real~~ $(N\times n)\times(N\times n)$ matrices of rank $N+M$.

$N+n = p + r$ $\quad$ $p = \#$ positive eigenv
$\quad$ $r = \#$ negative eigenv

$$\mathrm{Sgn}\,\Phi = p - r.$$

$$N+n - \mathrm{Sgn}\,\Phi = 2r.$$

$$\mathrm{Sgn}\,\Phi - \mathrm{Sgn}\,\tilde{\Phi} = 2(\tilde{r} - r)$$

$e^{i\frac{\pi}{4}(\mathrm{Sgn}\,\Phi - \mathrm{Sgn}\,\tilde{\Phi})}$ = Transition function of a line bundle on $\Lambda \equiv$ Maslov Bundle.

Similarly; if $\tilde{\varphi}(y, \tilde{\theta})$ parametrizes Λ near (x_0, y_0), then.

$$u(y) = \int e^{i\varphi(y, \tilde{\theta})} b(y, \tilde{\theta})\, d\tilde{\theta}.$$

and.

$$v(\xi)\, |d\xi|^{1/2} = a(x, \theta)\, e^{i\pi/4 \operatorname{sgn} \Phi}\, d_c^{1/2}$$

$$= b(\tilde{y}, \tilde{\theta})\, e^{i\pi/4 \operatorname{sgn} \tilde{\Phi}}\, d_{\tilde{c}}^{1/2}.$$

where $\partial_\theta \varphi = 0$, $\xi = \partial_x \varphi$

$\partial_{\tilde{\theta}} \tilde{\varphi} = 0$, $\xi = \partial_y \tilde{\varphi}$.

$$a(x, \theta)\, d_c^{1/2} = b(\tilde{y}, \tilde{\theta})\, d_{\tilde{c}}^{1/2}\, e^{i\pi/4(\operatorname{sgn} \tilde{\Phi} - \operatorname{sgn} \Phi)}$$