Transversality: Lecture 25

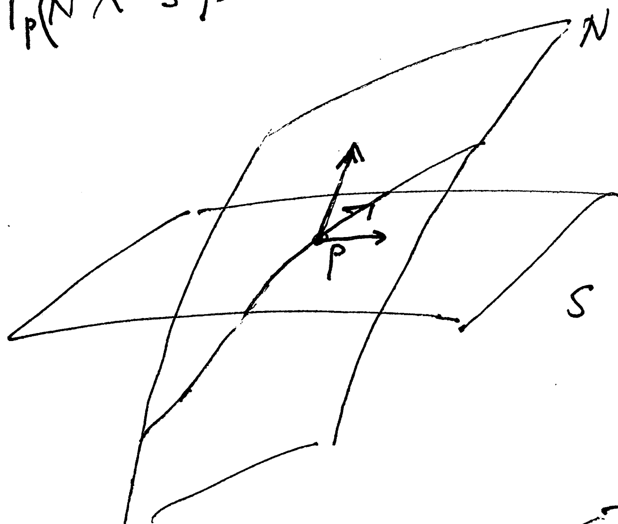
Let X be a C^∞ manifold and let $S \subset X$, $N \subset X$ be C^∞ submanifolds.

We say that S and N intersect transversally at p if

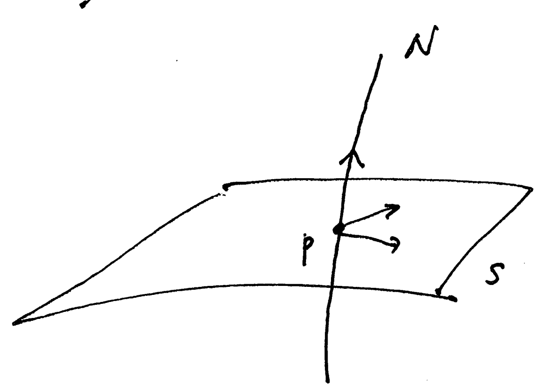
$$T_p S + T_p N = T_p X.$$

Proposition: If N and S intersect transversally ~~and~~ near p and at p $N \cap S$ is a C^∞ submanifold and

$$T_p(N \cap S) = T_p N \cap T_p S.$$



Proof:



The proposition follows from the following

Lemma: Under the hypotheses of the proposition there exist local coordinates $x = (x', x'')$

$x' = (x_1, \dots, x_n)$, $n = \dim X$ such that at

$p = 0$

$S = \{ x' = 0 \}$; $N = \{ x_{k+1} = \dots = x_\ell = 0 \}$

Proof: Particular case, - general case left as exercise.

$n = 3$:

$S = \{ y_1 = 0 \}$. If $\dim N = 2$ $N = \{ f = 0 \}$

In this case at least one ~~either~~ $\frac{\partial f}{\partial y_2}(0) \neq 0$ or $\frac{\partial f(0)}{\partial y_3}(0) \neq 0$

Suppose $\partial f / \partial y_2 \neq 0$

Set $x_1 = y_1$; $x_2 = f_2$, $x_3 = y_3$

$$\left| \frac{Dx}{Dy} \right| = \left| \det \begin{bmatrix} 1 & 0 & 0 \\ \frac{\partial f_2}{\partial y_1} & \frac{\partial f_2}{\partial y_2} & \frac{\partial f_2}{\partial y_3} \\ 0 & 0 & 1 \end{bmatrix} \right| = \left| \frac{\partial f_2}{\partial y_2}(0) \right| \neq 0.$$

$S = \{ y_1 = 0 \}$; $N = \{ f_1 = f_2 = 0 \}$

$T_p N = \operatorname{Ker} df_1(p) \cap \operatorname{Ker} df_2(p) = \operatorname{Span} \{ \partial_{x_1} \}$

This means that the ~~only~~ solutions to

$$df_1(p)\,V = 0 \quad = v_1 \frac{\partial f_1}{\partial y_1} + v_2 \frac{\partial f_1}{\partial y_2} + v_3 \frac{\partial f_1}{\partial y_3} = 0$$

$$df_2(p)\,V = 0 \quad = v_1 \frac{\partial f_2}{\partial y_1} + v_2 \frac{\partial f_2}{\partial y_2} + v_3 \frac{\partial f_2}{\partial y_3} = 0$$

satisfy $v_2 = v_3 = 0 \Rightarrow \det \begin{pmatrix} \frac{\partial f_1}{\partial y_2} & \frac{\partial f_1}{\partial y_3} \\ \frac{\partial f_2}{\partial y_2} & \frac{\partial f_2}{\partial y_3} \end{pmatrix} \neq 0$

Set $x_1 = y_1$, $x_2 = f_1$, $x_3 = f_2$.

$$\frac{Dx}{Dy} = \det \begin{pmatrix} 1 & 0 & 0 \\ \frac{\partial f_1}{\partial y_1} & \frac{\partial f_1}{\partial y_2} & \frac{\partial f_1}{\partial y_3} \\ \frac{\partial f_2}{\partial y_1} & \frac{\partial f_2}{\partial y_2} & \frac{\partial f_2}{\partial y_3} \end{pmatrix} = \det \begin{pmatrix} \frac{\partial f_1}{\partial y_2} & \frac{\partial f_1}{\partial y_3} \\ \frac{\partial f_2}{\partial y_2} & \frac{\partial f_2}{\partial y_3} \end{pmatrix} \neq 0$$

Remark: Clean Intersection: We say that N and S intersect cleanly at p if

$$T_p(N \cap S) = T_p N \cap T_p S$$

$N \cap S$ is a C^∞ manifold.

In this case one can show that there exist local coords $x = (x', x'', x''')$ such that

$$S = \{x' = 0,\ x'' = 0\}; \quad N = \{x' = 0,\ x_{k+1} = \dots = x_\ell = 0\}$$

$$x' = (x_j, x_{j+1}, \dots, x_n).$$

<u>Proposition:</u> Let X, Y, Z be C^∞ manifolds and $C_1 \subset (T^*X \backslash 0) \times (T^*Y \backslash 0)$

$C_2 \subset (T^*Y \backslash 0) \times (T^*Z \backslash 0)$ be homogeneous

canonical relations. Suppose that C_1 (C_1')

is parametrized by $\varphi(x, y, \theta)$ near (x_0, y_0, θ_0), $\theta \in \mathbb{R}^r$

and C_2 (C_2') is parametrized by $\psi(y, z, \tau)$, $\tau \in \mathbb{R}^m$

near (y_0, z_0, τ_0). If

$C_1 \times C_2 \pitchfork \tilde{D}$, where

$\tilde{D} = T^*X \backslash 0 \times \mathrm{Diag}\,(T^*Y \backslash 0 \times T^*Y \backslash 0) \times T^*Z \backslash 0$

~~near~~ at the corresponding points. Then

$C_1 \circ C_2$ is a C^∞ Lagrangian parametrized

by $\Phi(x, z, y, \theta, \tau) = \varphi(x, y, \theta) + \psi(y, z, \tau)$

when (y, θ, τ) are now regarded as parameters.

Proof: φ parametrizes C_1' and ψ parametrizes C_2' so. let

$$M = \{ (x, y, \theta) : \varphi'_\theta = 0 \}$$

$$N = \{ (y, z, \tau) : \psi'_\tau = 0 \}$$

$$C_1 = \{ (x, y, \xi, \eta) : \varphi'_\theta = 0 \,,\ \xi = \varphi'_x \,;\ \eta = -\varphi'_y \}$$

$$C_2 = \{ (y, \eta, z, \zeta) : \psi'_\tau = 0 \,,\ \eta = \psi'_y \,;\ \zeta = -\psi'_z \}$$

Then.

$$(C_1 \times C_2) \cap \tilde{D} = \{ (x, \xi, y', \eta' ; y'', \eta'', z, \zeta) \in C_1 \times C_2 : y' = y'' ;\ \eta' = \eta'' \} =$$

$$= \{ \varphi'_\theta = 0 \,,\ \psi'_\tau = 0 \,;\ \varphi'_y + \psi'_y = 0 \,;\ \xi = \varphi'_x \,,\ \zeta = -\psi'_z \}$$

Since the intersection is transversal, it means that $T_p\left((C_1 \times C_2) \cap \tilde{D} \right) =$

$$= T_p (C_1 \times C_2) \cap T_p \tilde{D}$$

or in other words that.

$$d_{(x,y,\theta)}\varphi'_\theta \; ; \quad d_{(y,z,\tau)}\psi'_\tau \; ; \quad d_{(x,y,\theta,y,z,\tau)}(\varphi'_y+\psi'_y)$$

$d_{(x,y,\theta)}\varphi'_x$ and $d_{(y,z,\tau)}\psi''_z$ are linearly independent and the tangent space of the intersection is the intersection of their kernels.

This implies that the ~~sole~~. function.

$$\Phi(x,z;\ y,\tau,\theta) = \varphi(x,y,\theta) + \psi(y,z,\tau)$$

is a non-degenerate phase function if one views (y,τ,θ) as parameters and

$$C_1 \circ C_2 = \{\varphi'_\theta = 0,\ \psi'_\tau = 0,\ \varphi'_y + \psi'_y = 0,$$
$$\xi = \varphi'_x\ ;\ \zeta = -\psi'_z\}$$

is a C^∞ manifold parametrized by Φ.

<u>Examples:</u> Let $\chi : T^*Y\setminus 0 \longrightarrow T^*X\setminus 0$

$$(y,\eta) \longmapsto (x,\xi)$$

be a symplectomorphism. $\chi^*\omega_X = \sigma_Y$.

$$C_\chi = \{ (x,\xi, y, \eta) : (x,\xi) = \chi(y,\eta) \}$$

C_χ is a homogeneous canonical relation in $T^*X\setminus 0 \times T^*Y\setminus 0$.

Let $C_2 \subset (T^*Y\setminus 0) \times (T^*Z\setminus 0)$

be a homogeneous canonical relation.

Claim $C_\chi \times C_2 \pitchfork \tilde{D}$.

Tangent space to $C_\chi = (d\chi V, V)$ $V \in T_p(T^*Y)$

Tangent space to $C_2 = (W, \tilde{W})$ with

Some relation between W and $\tilde{W}$.

$\tilde{D} = T^*X\setminus 0 \times (\mathrm{Diag}\,(T^*Y\setminus 0) \times (T^*X\setminus 0)) \times T^*Z\setminus 0$

Tangent space to $\tilde{D} = (A, -U, U, B)$.

Claim: Given (u_1, u_2, v_1, v_2) we can always
$w, \tilde{w}$

find

A, B, V, v such that

$$A + d\chi V = u_1$$

$$V - v = u_2.$$

$$W + U = v_1$$

$$\tilde{W} + B = v_2.$$

~~$B = v_2 - \tilde{w}$~~

$B = v_2 - \tilde{w}$; $v = v_1 - w$.

$$V = u_2 + v$$

then $A = u_1 - d\chi V$.

The key here is that V is arbitrary. We are using that C_χ is the graph of a canonical transformation.