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On a Class of Higher-Order Fully Decoupled Schemes for the Cahn–Hilliard–Navier–Stokes System

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Abstract
We construct a new class of fully decoupled and higher-order implicit-explicit schemes for the Cahn–Hilliard–Navier–Stokes System, which is a phase-field model of two-phase incompressible flows, based on the generalized scalar auxiliary variable approach with the new relaxation for the Cahn–Hilliard equation and the consistent splitting method for the Navier–Stokes equations. These schemes are linear, fully decoupled, only require solving a sequence of elliptic equations with constant coefficients at each time step. We show that numerical solutions of these schemes are uniformly bounded without any restriction on time step size. Furthermore, we carry out a rigorous error analysis for the first-order scheme and establish optimal global-in-time error estimates for the phase function, velocity and pressure in two and three-dimensions. Several numerical examples are presented to validate the accuracy and robustness of the proposed schemes.

Keywords Cahn–Hilliard · Navier–Stokes · Fully decoupled · Higher-order · Energy stability · Error estimates

Mathematics Subject Classification 65M12 · 65M15 · 35G31 · 65Z05

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1 Introduction

We consider in this paper numerical approximation of the following Cahn–Hilliard–Navier–Stokes system

$$\frac{\partial \phi}{\partial t} + (\mathbf{u} \cdot \nabla)\phi = M\Delta\mu \quad in\ \Omega \times J, \tag{1.1a}$$

$$\mu = -\lambda\Delta\phi + \lambda G'(\phi) \quad in\ \Omega \times J, \tag{1.1b}$$

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla\mathbf{u} - \nu\Delta\mathbf{u} + \nabla p = \mu\nabla\phi \quad in\ \Omega \times J, \tag{1.1c}$$

$$\nabla \cdot \mathbf{u} = 0 \quad in\ \Omega \times J, \tag{1.1d}$$

$$\frac{\partial \phi}{\partial \mathbf{n}} = \frac{\partial \mu}{\partial \mathbf{n}} = 0,\ \ \mathbf{u} = \mathbf{0} \quad on\ \partial\Omega \times J, \tag{1.1e}$$

where $G(\phi) = \frac{1}{4\epsilon^2}(1-\phi^2)^2$ with ϵ representing the interfacial width, $M > 0$ is the mobility constant, $\lambda > 0$ is the mixing coefficient, $\nu > 0$ is the fluid viscosity. Ω is a bounded domain in $\mathbb{R}^d$ $(d = 2, 3)$ and $J = (0, T]$. The unknowns are the velocity $\mathbf{u}$, the pressure p, the phase function ϕ and the chemical potential μ. We refer to [11, 20] for its physical interpretation and derivation as a phase-field model for the incompressible two phase flow with matching density (set to be $\rho_0 = 1$ for simplicity), and to [1] for its mathematical analysis. The above system satisfies the following energy dissipation law:

$$\frac{dE(\phi, \mathbf{u})}{dt} = -M\|\nabla\mu\|^2 - \nu\|\nabla\mathbf{u}\|^2 \ \text{ with } \ E(\phi, \mathbf{u}) = \int_\Omega \left\{ \frac{1}{2}|\mathbf{u}|^2 + \frac{\lambda}{2}|\nabla\phi|^2 + \lambda G(\phi) \right\} d\mathbf{x}. \tag{1.2}$$

It is crucial that numerical schemes for (1.1) preserve a dissipative energy law at the discrete level.

Various energy stable numerical methods have been proposed for Navier–Stokes equations and for Cahn–Hilliard equations. The main issue in dealing with the Navier–Stokes equation is the coupling of velocity and pressure by the incompressible condition $\nabla \cdot \mathbf{u} = 0$. A partial list of earlier works includes those three categories [9]: the pressure-correction method [19, 24], the velocity-correction method [10, 22] and the consistent splitting method [8] (see also the gauge method [32]). Among these, the consistent splitting scheme has great advantages in two aspects: (i) this method can achieve full accuracy of the time discretization since it is not limited by splitting error; (ii) The inf-sup condition between the velocity and the pressure approximation spaces is no longer enforced from a computational point of view. A main difficulty in solving the Cahn–Hilliard equation is how to deal with the nonlinear term efficiently so that the resulting system can be effectively solved while preserving an energy dissipation law. There are several popular approaches including the convex splitting method [5], stabilized semi-implicit method [27], invariant energy quadratization (IEQ) [31], and scalar auxiliary variable (SAV) [26]. For an up-to-date review on various classical methods for gradient flows especially for the Cahn–Hilliard equation, one can refer to [4, 30].

There are also many studies devoted to developing efficient numerical schemes and carrying out corresponding error analysis for the Cahn–Hilliard–Navier–Stokes phase-field models [28, 29]. Fully coupled first-order-in-time implicit semi-discrete and fully discrete finite element schemes are considered by Feng, He and Liu [6] and convergence results are established rigorously. Grün [7] established an abstract convergence result for a fully discrete implicit scheme for diffuse interface models of two-phase incompressible fluids with different densities. A coupled second-order energy stable scheme for the Cahn–Hilliard–Navier–Stokes system based on convex splitting for the Cahn–Hilliard equation is constructed by Han and Wang [13]. In addition, Han et al. [12] developed a class of second-order IEQ schemes which can preserve energy stability. In 2020, we [17] constructed a second-order weakly-coupled, linear, energy stable SAV-MAC scheme for the Cahn–Hilliard–Navier–Stokes equations, and established second order convergence both in time and space for the simpler Cahn–Hilliard-Stokes equations.

It is important to note that all of the aforementioned works involve solving a coupled linear or nonlinear system with variable coefficients at each time step. Recently in [18], we construct first- and second-order time discretization schemes for the Cahn–Hilliard–Navier–Stokes system based on the MSAV approach for gradient systems and (rotational) pressure-correction for Navier–Stokes equations. These schemes are linear, fully decoupled, unconditionally energy stable, and only require solving a sequence of elliptic equations with constant coefficients at each time step. In the above work, we only established error estimates for two-dimensional case, as the weak stability results established there were not sufficient for establishing an error estimate in 3D. It is also much more difficult to construct higher than second-order fully decoupled numerical schemes due to the splitting error of the pressure-correction method.

The main purposes of this work are to construct a class of higher-order fully decoupled, linear and unconditionally energy stable schemes for (1.1), and to carry out a rigorous error analysis in two and three-dimensional cases. Our main contributions are:

- By using a combination of techniques in the GSAV approach [15] with the new relaxation and the consistent splitting method [8, 16], we construct new fully decoupled, linear and higher-order schemes for the Cahn–Hilliard–Navier–Stokes system, which only require solving a sequence of Poisson type equations with constant coefficients at each time step and are unconditionally energy stable with a modified energy that is directly linked to the original energy.
- We establish global-in-time error estimates in $l^\infty(0, T; H^1(\Omega)) \bigcap l^2(0, T; H^2(\Omega))$ for the velocity and $l^2(0, T; H^1(\Omega))$ for the pressure, and $l^\infty(0, T; H^1(\Omega))$ for the phase function in two and three-dimensional cases.

We believe that our higher-order, fully decoupled, linear, unconditionally energy stable scheme is the first such scheme for the Cahn–Hilliard–Navier–Stokes system, and its global-in-time error analysis in the three-dimensional case is the first for any linear and fully decoupled schemes with explicit treatment of all nonlinear terms.

The paper is organized as follows. In Sect. 2, we provide some preliminaries which will be used in the sequel. In Sect. 3, we construct the fully decoupled consistent splitting GSAV schemes and prove that they are unconditionally energy stable with a modified energy. In Sect. 4, we carry out an error analysis for the first-order consistent splitting GSAV scheme. In Sect. 5, we present numerical experiments to validate our proposed schemes.

2 Preliminaries

We introduce some standard notations. Let $L^m(\Omega)$ be the standard Banach space with norm

$$\|v\|_{L^m(\Omega)} = \left(\int_\Omega |v|^m d\Omega\right)^{1/m}.$$

For the case $m = \infty$, set $\|v\|_\infty = \|v\|_{L^\infty(\Omega)} = ess\ sup\{|f(x)| : x \in \Omega\}$. And $W^{k,p}(\Omega)$ be the standard Sobolev space

$$W^{k,p}(\Omega) = \{g : \ \|g\|_{W_p^k(\Omega)} < \infty\},$$

where

$$\|g\|_{W^{k,p}(\Omega)} = \left(\sum_{|\alpha| \le k} \|D^\alpha g\|_{L^p(\Omega)}^p \right)^{1/p}, \tag{2.1}$$

in the case $1 \le p < \infty$, and in the case $p = \infty$,

$$\|g\|_{W^{k,\infty}(\Omega)} = \max_{|\alpha| \le k} \|D^\alpha g\|_{L^\infty(\Omega)}.$$

For simplicity, we set $H^k(\Omega) = W^{k,2}(\Omega)$ and $\|f\|_k = \|f\|_{H^k(\Omega)}$.

By using Poincaré inequality, we have

$$\|\mathbf{v}\| \le c_1 \|\nabla \mathbf{v}\|, \ \forall\, \mathbf{v} \in \mathbf{H}_0^1(\Omega), \tag{2.2}$$

where c_1 is a positive constant depending only on Ω and

$$\mathbf{H}_0^1(\Omega) = \left\{ \mathbf{v} \in \mathbf{H}^1(\Omega) : \mathbf{v}|_\Gamma = 0 \right\}.$$

Define

$$\mathbf{H} = \left\{ \mathbf{v} \in \mathbf{L}^2(\Omega) : div\mathbf{v} = 0, \mathbf{v} \cdot \mathbf{n}|_\Gamma = 0 \right\}, \quad \mathbf{V} = \left\{ \mathbf{v} \in \mathbf{H}_0^1(\Omega) : div\mathbf{v} = 0 \right\},$$

and the trilinear form $b(\cdot, \cdot, \cdot)$ by

$$b(\mathbf{u}, \mathbf{v}, \mathbf{w}) = \int_\Omega (\mathbf{u} \cdot \nabla)\mathbf{v} \cdot \mathbf{w} d\mathbf{x}.$$

We can easily observe that the trilinear form $b(\cdot, \cdot, \cdot)$ is skew-symmetric with respect to its last two arguments, i.e.,

$$b(\mathbf{u}, \mathbf{v}, \mathbf{w}) = -b(\mathbf{u}, \mathbf{w}, \mathbf{v}), \quad \forall\, \mathbf{u} \in \mathbf{H}, \quad \mathbf{v}, \mathbf{w} \in \mathbf{H}_0^1(\Omega), \tag{2.3}$$

$$b(\mathbf{u}, \mathbf{v}, \mathbf{v}) = 0, \quad \forall\, \mathbf{u} \in \mathbf{H}, \quad \mathbf{v} \in \mathbf{H}_0^1(\Omega). \tag{2.4}$$

The following lemmas will be frequently used in the sequel, one can refer the proof and more detailed information in [2, 3]:

Lemma 2.1 (Holder inequality) *Suppose that* $\boldsymbol{u} \in \boldsymbol{L}^p(\Omega)$, $\boldsymbol{v} \in \boldsymbol{L}^q(\Omega)$, $w \in L^s(\Omega)$, $\frac{1}{p} + \frac{1}{q} + \frac{1}{s} = 1$, *then we have*

$$\int_\Omega |(\boldsymbol{u}, \boldsymbol{v}) w| d\boldsymbol{x} \le \|\boldsymbol{u}\|_{\boldsymbol{L}^p} \|\boldsymbol{v}\|_{\boldsymbol{L}^q} \|w\|_{L^s}. \tag{2.5}$$

Lemma 2.2 (Interpolation inequalities) *For* $k = 3, 4, 6$, *we have*

$$\|\boldsymbol{f}\|_{\boldsymbol{L}^k} \le C \|\boldsymbol{f}\|_{\boldsymbol{L}^2}^{\frac{6-k}{2k}} \|\boldsymbol{f}\|_{\boldsymbol{H}^1}^{\frac{3k-6}{2k}}, \quad \|\boldsymbol{f}\|_{\boldsymbol{L}^\infty} \le C \|\boldsymbol{f}\|_{\boldsymbol{H}^1}^{\frac{1}{2}} \|\boldsymbol{f}\|_{\boldsymbol{H}^2}^{\frac{1}{2}}. \tag{2.6}$$

We will frequently use the following discrete version of the Grönwall lemma [23]:

Lemma 2.3 *Let a_k, b_k, c_k, d_k, γ_k, Δt_k be nonnegative real numbers such that*

$$a_{k+1} - a_k + b_{k+1}\Delta t_{k+1} + c_{k+1}\Delta t_{k+1} - c_k\Delta t_k \le a_k d_k \Delta t_k + \gamma_{k+1}\Delta t_{k+1} \tag{2.7}$$

for all $0 \le k \le m$. Then

$$a_{m+1} + \sum_{k=0}^{m+1} b_k \Delta t_k \le \exp\left(\sum_{k=0}^{m} d_k \Delta t_k\right)\{a_0 + (b_0 + c_0)\Delta t_0 + \sum_{k=1}^{m+1}\gamma_k\Delta t_k\}. \tag{2.8}$$

3 The GSAV Scheme with the New Relaxation

In this section, we first reformulate the Cahn–Hilliard–Navier–Stokes system into an equivalent system with generalized scalar auxiliary variables (GSAV). Then, we construct higher-order fully decoupled semi-discrete consistent splitting GSAV schemes with the new relaxation based on the IMEX BDF-k formulae with $k = 1, 2, 3, 4, 5$, and prove that they are unconditionally energy stable.

3.1 GSAV Reformulation

Let $\gamma > 0$ be a positive constant, $F(\phi) = G(\phi) - \frac{\gamma}{2}\phi^2$ and

$$E(\phi, \mathbf{u}) = \int_\Omega \left\{\frac{1}{2}|\mathbf{u}|^2 + \frac{\lambda}{2}|\nabla\phi|^2 + \frac{\lambda\gamma}{2}\phi^2 + \lambda F(\phi)\right\} d\mathbf{x}, \tag{3.1}$$

where the term $\frac{\lambda\gamma}{2}\phi^2$ is introduced to simplify the analysis (cf. [25]). We introduce the following generalized scalar auxiliary variable

$$r(t) = E(\phi, \mathbf{u}) + \kappa_0, \tag{3.2a}$$

where κ_0 is a positive constant to guarantee that $\lambda\int_\Omega F(\phi)d\mathbf{x} + \kappa_0 > 0$. Next we reformulate the system (1.1) as:

$$\frac{\partial\phi}{\partial t} + (\mathbf{u}\cdot\nabla)\phi = M\Delta\mu \quad in\ \Omega\times J, \tag{3.3a}$$

$$\mu = -\lambda\Delta\phi + \lambda\gamma\phi + \lambda F'(\phi) \quad in\ \Omega\times J, \tag{3.3b}$$

$$\frac{\partial\mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u} - \nu\Delta\mathbf{u} + \nabla p = \mu\nabla\phi \quad in\ \Omega\times J, \tag{3.3c}$$

$$\nabla\cdot\mathbf{u} = 0 \quad in\ \Omega\times J, \tag{3.3d}$$

$$\frac{\mathrm{d}r}{\mathrm{d}t} = -M\|\nabla\mu\|^2 - \nu\|\nabla\mathbf{u}\|^2 \quad in\ \Omega\times J. \tag{3.3e}$$

It is easy to see that the above system is equivalent to the original system. We shall construct below efficient numerical schemes for the above system which are energy stable with respect to (3.3e).

Assuming $\tilde{\phi}^j$, $\tilde{\mu}^j$ and $\tilde{\mathbf{u}}^j$ with $j = n, n-1, \ldots, n-l+1$ are given, we fist solve $\tilde{\phi}^{n+1}$ and $\tilde{\mu}^{n+1}$ from

$$\frac{\alpha_k\tilde{\phi}^{n+1} - A_k(\tilde{\phi}^n)}{\Delta t} + (B_k(\mathbf{u}^n)\cdot\nabla)B_k(\phi^n) = M\Delta\tilde{\mu}^{n+1}, \quad \frac{\partial\phi}{\partial\mathbf{n}}|_{\partial\Omega} = 0, \tag{3.4}$$

$$\tilde{\mu}^{n+1} = -\lambda\Delta\tilde{\phi}^{n+1} + \lambda\gamma\tilde{\phi}^{n+1} + \lambda F'(B_k(\phi^n)), \quad \frac{\partial\mu}{\partial\mathbf{n}}|_{\partial\Omega} = 0. \tag{3.5}$$

Then, we solve $\tilde{\mathbf{u}}^{n+1}$ from

$$\begin{aligned}&\frac{\alpha_k\tilde{\mathbf{u}}^{n+1}-A_k(\tilde{\mathbf{u}}^n)}{\Delta t}-\nu\Delta\tilde{\mathbf{u}}^{n+1}=B_k(\mu^n)\nabla B_k(\phi^n)-(B_k(\mathbf{u}^n)\cdot\nabla)B_k(\mathbf{u}^n)-\nabla B_k(p^n),\\&\tilde{\mathbf{u}}^{n+1}|_{\partial\Omega}=0.\end{aligned} \tag{3.6}$$

For readers' convenience, α_k, the operators A_k and B_k with $k=1,2,3,4,5$ are given below:

first-order scheme:

$$\alpha_1=1,\ A_1(f^n)=f^n,\ B_1(g^n)=g^n;$$

second-order scheme:

$$\alpha_2=\frac{3}{2},\ A_2(f^n)=2f^n-\frac{1}{2}f^{n-1},\ B_2(g^n)=2g^n-g^{n-1};$$

third-order scheme:

$$\alpha_3=\frac{11}{6},\ A_3(f^n)=3f^n-\frac{3}{2}f^{n-1}+\frac{1}{3}f^{n-2},\ B_3(g^n)=3g^n-3g^{n-1}+g^{n-2}.$$

fourth-order scheme:

$$\begin{aligned}\alpha_4=\frac{25}{12},\ A_4(f^n)&=4f^n-3f^{n-1}+\frac{4}{3}f^{n-2}-\frac{1}{4}f^{n-3},\\B_4(g^n)&=4g^n-6g^{n-1}+4g^{n-2}-g^{n-3};\end{aligned}$$

fifth-order scheme:

$$\begin{aligned}\alpha_5=\frac{137}{60},\ A_5(f^n)&=5f^n-5f^{n-1}+\frac{10}{3}f^{n-2}-\frac{5}{4}f^{n-3}+\frac{1}{5}f^{n-4},\\B_5(g^n)&=5g^n-10g^{n-1}+10g^{n-2}-5g^{n-3}+g^{n-4}.\end{aligned}$$

Then we solve $\tilde{R}^{n+1}$, ξ^{n+1} from

$$\frac{\tilde{R}^{n+1}-R^n}{\Delta t}=-\xi^{n+1}\left(M\|\nabla\tilde{\mu}^{n+1}\|^2+\nu\|\nabla B_k(\mathbf{u}^n)\|^2\right),\quad \xi^{n+1}=\frac{\tilde{R}^{n+1}}{E(\tilde{\phi}^{n+1},\tilde{\mathbf{u}}^{n+1})+\kappa_0}. \tag{3.7}$$

Next we update ϕ^{n+1}, μ^{n+1}, $\mathbf{u}^{n+1}$ by

$$\phi^{n+1}=\eta_k^{n+1}\tilde{\phi}^{n+1},\quad \mu^{n+1}=\eta_k^{n+1}\tilde{\mu}^{n+1},\quad \mathbf{u}^{n+1}=\eta_k^{n+1}\tilde{\mathbf{u}}^{n+1}, \tag{3.8}$$

where

$$\eta_1^{n+1}=1-(1-\xi^{n+1})^2;\ \eta_k^{n+1}=1-(1-\xi^{n+1})^k\quad \text{for } k=2,3,4,5. \tag{3.9}$$

Then we update the SAV R^{n+1} as

$$R^{n+1}=\min\left\{R^n, E(\phi^{n+1},\mathbf{u}^{n+1})+\kappa_0\right\}. \tag{3.10}$$

Finally, we determine p^{n+1} by solving

$$(\nabla p^{n+1},\nabla q)=\left(\mu^{n+1}\nabla\phi^{n+1}-(\mathbf{u}^{n+1}\cdot\nabla)\mathbf{u}^{n+1}-\nu\nabla\times\nabla\times\tilde{\mathbf{u}}^{n+1},\nabla q\right),\ \forall q\in H^1(\Omega). \tag{3.11}$$

Remark 3.1 The above scheme is easy to implement and very efficient. Indeed, (3.4)–(3.5) is a coupled second-order equations with constant coefficients; (3.6) is a second-order equation with constant coefficients; (3.11) is a Poisson equation in the weak form; (3.7)–(3.10) can be updated directly.

Next we prove the following unconditional energy stability with a modified energy for the above schemes (3.4)–(3.11):

Theorem 3.1 *Given $R^n > 0$, Then for (3.4)–(3.10), we have $\xi^{n+1} > 0$ and*

$$0 < R^{n+1} \le R^n, \quad \forall n \le T/\Delta t. \tag{3.12}$$

In addition, there exists a constant M_T independent of Δt such that

$$\|\boldsymbol{u}^{n+1}\|^2 + \lambda\|\nabla\phi^{n+1}\|^2 + \lambda\gamma\|\phi^{n+1}\|^2 \le M_T, \quad \forall n \le T/\Delta t. \tag{3.13}$$

Proof Given $R^n > 0$, it follows from (3.10) that (3.12) holds.

In addition, noting that $\tilde{R}^0 = R^0$. It follows from (3.7) that

$$0 < \tilde{R}^{n+1} = \frac{1}{1 + \Delta t \frac{M\|\nabla\tilde{\mu}^{n+1}\|^2 + \nu\|\nabla B_k(\mathbf{u}^n)\|^2}{E(\tilde{\phi}^{n+1}, \tilde{\mathbf{u}}^{n+1}) + \kappa_0}} R^n < R^n, \tag{3.14}$$

then we have $\xi^{n+1} > 0$. Denote $R^0 := M$, it then follows from (3.1) and (3.14) that

$$\xi^{n+1} = \frac{\tilde{R}^{n+1}}{E(\tilde{\phi}^{n+1}, \tilde{\mathbf{u}}^{n+1}) + \kappa_0} \le \frac{2M}{\|\tilde{\mathbf{u}}^{n+1}\|^2 + \lambda\|\nabla\tilde{\phi}^{n+1}\|^2 + \lambda\gamma\|\tilde{\phi}^{n+1}\|^2 + 2}, \tag{3.15}$$

where without loss of generality, we assume the positive constant $\int_\Omega F(\phi)d\mathbf{x} + \kappa_0 > 1$. Recalling (3.9), we can derive from (3.15) that there exists a positive constant M_1 such that

$$|\eta_l^{n+1}| = |\xi^{n+1} P_q(\xi^{n+1})| \le \frac{M_1}{\|\tilde{\mathbf{u}}^{n+1}\|^2 + \lambda\|\nabla\tilde{\phi}^{n+1}\|^2 + \lambda\gamma\|\tilde{\phi}^{n+1}\|^2 + 2}, \tag{3.16}$$

where P_q is a polynomial function of degree q with $q = 1$ for $l = 1$ and $q = l - 1$ for $l = 2, 3, 4, 5$. Thus we have

$$\begin{aligned} \|\mathbf{u}^{n+1}\|^2 + \lambda\|\nabla\phi^{n+1}\|^2 + \lambda\gamma\|\phi^{n+1}\|^2 &= (\eta_l^{n+1})^2(\|\tilde{\mathbf{u}}^{n+1}\|^2 + \lambda\|\nabla\tilde{\phi}^{n+1}\|^2 + \lambda\gamma\|\tilde{\phi}^{n+1}\|^2) \\ &\le \left(\frac{M_1}{\|\tilde{\mathbf{u}}^{n+1}\|^2 + \lambda\|\nabla\tilde{\phi}^{n+1}\|^2 + \lambda\gamma\|\tilde{\phi}^{n+1}\|^2 + 2}\right)^2 \\ &\quad (\|\tilde{\mathbf{u}}^{n+1}\|^2 + \lambda\|\nabla\tilde{\phi}^{n+1}\|^2 + \lambda\gamma\|\tilde{\phi}^{n+1}\|^2) \\ &\le M_1^2, \end{aligned} \tag{3.17}$$

which implies the desired results (3.13). □

Now we shall first establish the relation between $\tilde{R}^{k+1}$ and R^{k+1} to give following estimates.

Lemma 3.2 *Using the definition (3.10), we have*

$$R^{k+1} = \sigma^{k+1}\tilde{R}^{k+1} + (1 - \sigma^{k+1})(E(\phi^{k+1}, \boldsymbol{u}^{k+1}) + \kappa_0), \tag{3.18}$$

where

$$\begin{cases} \sigma^{k+1} = 0, \ \text{if } R^k \ge E(\phi^{k+1}, \boldsymbol{u}^{k+1}) + \kappa_0, \\ \sigma^{k+1} = 1 - \frac{\tilde{R}^{k+1}\left(M\|\nabla\tilde{\mu}^{k+1}\|^2 + \nu\|\nabla B_l(\boldsymbol{u}^k)\|^2\right)}{(E(\tilde{\phi}^{k+1}, \tilde{\boldsymbol{u}}^{k+1}) + \kappa_0)(E(\phi^{k+1}, \boldsymbol{u}^{k+1}) + \kappa_0 - \tilde{R}^{k+1})}\Delta t, \ \text{if } R^k < E(\phi^{k+1}, \boldsymbol{u}^{k+1}) + \kappa_0. \end{cases} \tag{3.19}$$

Proof Recalling (3.10), it is easy to obtain that if $R^k \geq E(\phi^{k+1}, \mathbf{u}^{k+1}) + \kappa_0$,

$$R^{k+1} = \min\left\{R^k, E(\phi^{k+1}, \mathbf{u}^{k+1}) + \kappa_0\right\} = E(\phi^{k+1}, \mathbf{u}^{k+1}) + \kappa_0. \tag{3.20}$$

Thus we can easily obtain that $\sigma^{k+1} = 0$ in (3.18).

On the other hand, if $R^k < E(\phi^{k+1}, \mathbf{u}^{k+1}) + \kappa_0$, we have $R^{k+1} = R^k$. Thus by using (3.14), we have

$$\tilde{R}^{k+1} < R^k = R^{k+1} < E(\phi^{k+1}, \mathbf{u}^{k+1}) + \kappa_0. \tag{3.21}$$

Thus there exists a constant $0 < \sigma^{k+1} < 1$ to satisfy (3.18). Using (3.14) leads to

$$\begin{aligned} R^{k+1} =& R^k = \left(1 + \Delta t \frac{M\|\nabla\tilde{\mu}^{k+1}\|^2 + \nu\|\nabla B_l(\mathbf{u}^k)\|^2}{E(\tilde{\phi}^{k+1}, \tilde{\mathbf{u}}^{k+1}) + \kappa_0}\right)\tilde{R}^{k+1} \\ =& \sigma^{k+1}\tilde{R}^{k+1} + (1-\sigma^{k+1})(E(\phi^{k+1}, \mathbf{u}^{k+1}) + \kappa_0), \end{aligned} \tag{3.22}$$

which implies that

$$\sigma^{k+1} = 1 - \frac{\tilde{R}^{k+1}\left(M\|\nabla\tilde{\mu}^{k+1}\|^2 + \nu\|\nabla B_l(\mathbf{u}^k)\|^2\right)}{(E(\tilde{\phi}^{k+1}, \tilde{\mathbf{u}}^{k+1}) + \kappa_0)(E(\phi^{k+1}, \mathbf{u}^{k+1}) + \kappa_0 - \tilde{R}^{k+1})}\Delta t.$$

The proof is complete. □

4 Error Analysis

We note that even in the case of linear time dependent Stokes equations, the stability of second- and higher-order consistent splitting scheme based on the usual BDF is still an open problem. The main difficulty lies in the fact that the coefficient of the estimate for commutator of the Laplacian and Leray-Helmholtz projection operators is not sufficiently small, which means that the extrapolation of this term to a higher order poses considerable difficulties for theoretical analysis. So here we only carry out an error analysis for the scheme (3.4)–(3.11) in the first-order case. The recent progress in [16] based on a generalized BDF offers potential for an error analysis of the scheme (3.4)–(3.11) at higher-order discretization, but it will be much more involved and beyond the scope of this paper.

For purely technical reasons, we shall modify the definition of R^{k+1} in (3.10) slightly to (3.18) with

$$\begin{cases} \sigma^{k+1} = 0, \text{ if } R^k \geq E(\phi^{k+1}, \mathbf{u}^{k+1}) + \kappa_0, \\ \sigma^{k+1} = 1 - \frac{\tilde{R}^{k+1}\left(M\|\nabla\tilde{\mu}^{k+1}\|^2 + \nu\|\nabla B_l(\mathbf{u}^k)\|^2\right)}{(E(\tilde{\phi}^{k+1}, \tilde{\mathbf{u}}^{k+1}) + \kappa_0)(E(\phi^{k+1}, \mathbf{u}^{k+1}) + \kappa_0 - \tilde{R}^{k+1})}(\Delta t)^2, \text{ if } R^k < E(\phi^{k+1}, \mathbf{u}^{k+1}) + \kappa_0. \end{cases} \tag{4.1}$$

Note that Theorem 3.1 still holds with this modification if $\Delta t < 1$. Indeed, if $R^k < E(\phi^{k+1}, \mathbf{u}^{k+1}) + \kappa_0$, we obtain from (3.7) that $\tilde{R}^{k+1} \leq R^k$. Hence, we can write

$$R^k = \gamma^{k+1}\tilde{R}^{k+1} + (1-\gamma^{k+1})(E(\phi^{k+1}, \mathbf{u}^{k+1}) + \kappa_0), \tag{4.2}$$

and derive from (3.7) that $\gamma^{k+1} = 1 - H\Delta t$ with $H := \frac{\tilde{R}^{k+1}\left(M\|\nabla\tilde{\mu}^{k+1}\|^2 + \nu\|\nabla\mathbf{u}^k\|^2\right)}{(E(\tilde{\phi}^{k+1}, \tilde{\mathbf{u}}^{k+1}) + \kappa_0)(E(\phi^{k+1}, \mathbf{u}^{k+1}) + \kappa_0 - \tilde{R}^{k+1})}$. Hence we have

$$R^k = \tilde{R}^{k+1} - H\Delta t\tilde{R}^{k+1} + H\Delta t(E(\phi^{k+1}, \mathbf{u}^{k+1}) + \kappa_0). \tag{4.3}$$

On the other hand, we have from (4.1) that

$$\sigma^{k+1} = 1 - H(\Delta t)^2. \tag{4.4}$$

We then obtain from (3.18) that

$$\begin{aligned} R^{k+1} =& \sigma^{k+1}\tilde{R}^{k+1} + (1-\sigma^{k+1})(E(\phi^{k+1},\mathbf{u}^{k+1})+\kappa_0) \\ =& \tilde{R}^{k+1} - H(\Delta t)^2\tilde{R}^{k+1} + H(\Delta t)^2(E(\phi^{k+1},\mathbf{u}^{k+1})+\kappa_0). \end{aligned} \tag{4.5}$$

Subtracting (4.3) from (4.5), we have

$$R^{k+1} - R^k = H\Delta t(1-\Delta t)\left(\tilde{R}^{k+1} - (E(\phi^{k+1},\mathbf{u}^{k+1})+\kappa_0)\right), \tag{4.6}$$

which, together with $\tilde{R}^{k+1} \le R^k < (E(\phi^{k+1},\mathbf{u}^{k+1})+\kappa_0)$ implies

$$R^{k+1} < R^k, \quad \text{if } \Delta t < 1. \tag{4.7}$$

For notational simplicity, we shall drop the dependence on x for all functions when there is no confusion. Let $(\phi,\mu,\mathbf{u},p,r)$ be the exact solution of (3.3), and $(\phi^{n+1},\tilde{\phi}^{n+1},\mu^{n+1},\tilde{\mu}^{n+1},\tilde{\mathbf{u}}^{n+1},\mathbf{u}^{n+1},$
$p^{n+1},R^{n+1},\tilde{R}^{n+1})$ be the solution of the scheme (3.4–3.11), we denote

$$\begin{cases} \tilde{e}_\phi^{n+1} = \tilde{\phi}^{n+1} - \phi(t^{n+1}), & e_\phi^{n+1} = \phi^{n+1} - \phi(t^{n+1}), \\ \tilde{e}_\mu^{n+1} = \tilde{\mu}^{n+1} - \mu(t^{n+1}), & e_\mu^{n+1} = \mu^{n+1} - \mu(t^{n+1}), \\ \tilde{e}_\mathbf{u}^{n+1} = \tilde{\mathbf{u}}^{n+1} - \mathbf{u}(t^{n+1}), & e_\mathbf{u}^{n+1} = \mathbf{u}^{n+1} - \mathbf{u}(t^{n+1}), \\ \tilde{e}_R^{n+1} = \tilde{R}^{n+1} - r(t^{n+1}), & e_R^{n+1} = R^{n+1} - r(t^{n+1}). \\ e_p^{n+1} = p^{n+1} - p(t^{n+1}). \end{cases} \tag{4.8}$$

The main results are stated in the following theorem:

Theorem 4.1 *Assuming* $\phi \in W^{2,\infty}(0,T;L^2(\Omega))\bigcap W^{1,\infty}(0,T;H^1(\Omega))\bigcap L^\infty(0,T;H^2(\Omega))$, $\mu \in W^{1,\infty}(0,T;H^2(\Omega))$, $\boldsymbol{u} \in W^{1,\infty}(0,T;\boldsymbol{H}^2(\Omega))\bigcap W^{2,\infty}(0,T;\boldsymbol{L}^2(\Omega))$, *and* $p \in W^{1,\infty}(0,T;L^2(\Omega))$, *then for the first-order scheme* (3.4)-(3.11) *with* (3.10) *replaced by* (3.18) *with* (4.1), *we have*

$$\begin{aligned} \|e_\phi^{n+1}\|^2 + &\|\nabla e_\phi^{n+1}\|^2 + \Delta t\sum_{k=0}^n \|e_\mu^{k+1}\|^2 + \Delta t\sum_{k=0}^n \|\nabla e_\mu^{k+1}\|^2 \\ &+ \|\nabla e_{\boldsymbol{u}}^{n+1}\|^2 + \Delta t\sum_{k=0}^n \|\Delta e_{\boldsymbol{u}}^{k+1}\|^2 + \Delta t\sum_{k=0}^n \|\nabla e_p^{k+1}\|^2 \le C(\Delta t)^2, \ \forall n \le T/\Delta t, \end{aligned}$$

under the condition that $\Delta t \le \frac{1}{1+C_0^2}$, *where* C_0 *is independent of* Δt *and will be specified in what follows.*

Proof The proof of the above theorem will be carried out through a sequence of intermediate lemmas. First we shall make the hypothesis that there exists a positive constant C_0 such that

$$|1-\xi^k| \le C_0\Delta t, \ \forall k \le T/\Delta t, \tag{4.9}$$

$$\|\tilde{e}_\mathbf{u}^k\|_{H^2} + \|\tilde{e}_\mu^k\|_{H^1} \le (\Delta t)^{1/6}, \ \forall k \le T/\Delta t, \tag{4.10}$$

which will be proved in the induction process below by using a bootstrap argument.

We can easily obtain that (4.9) and (4.10) hold for $k=0$ by setting $\xi^0 = 1$. Now we suppose

$$|1-\xi^k| \le C_0\Delta t, \ \forall k \le n, \tag{4.11}$$

$$\|\tilde{e}_\mathbf{u}^k\|_{H^2} + \|\tilde{e}_\mu^k\|_{H^1} \le (\Delta t)^{1/6}, \ \forall k \le n, \tag{4.12}$$

and we shall prove that $|1-\xi^{n+1}| \le C_0\Delta t$ and $\|\tilde{e}_{\mathbf{u}}^{n+1}\|_{H^2} + \|\tilde{e}_{\mu}^{n+1}\|_{H^1} \le (\Delta t)^{1/6}$ hold true.
Step 1: Estimates for H^1 bounds of $\|\tilde{e}_{\phi}^{n+1}\|$. First using exactly the same procedure in [14], we can easily obtain that

$$\frac{1}{2} \le |\xi^k|,\ |\eta^k| \le 2, \tag{4.13}$$

under the condition $\Delta t \le \min\{\frac{1}{4C_0}, 1\}$.

We shall first derive an $H^2(\Omega)$ bound for ϕ^n without assuming the Lipschitz condition on $F(\phi)$. A key ingredient is the following stability result

$$\|\mathbf{u}^{n+1}\|^2 + \|\phi^{n+1}\|_{H^1}^2 + |R^{n+1}|^2 \le K_1, \tag{4.14}$$

where the positive constant K_1 is dependent on $\mathbf{u}^0$ and ϕ^0, which can be derived from the unconditionally energy stability (3.13). □

Lemma 4.2 *Under the assumption of Theorem 4.1, there exists a positive constant K_2 independent of Δt such that*

$$\|\Delta\tilde{\phi}^{k+1}\|^2 + \|\tilde{\mu}^{k+1}\|^2 + \sum_{k=0}^{n} \Delta t \|\nabla\tilde{\mu}^{k+1}\|^2 \le K_2,\ \forall\, 0 \le k \le n+1.$$

Proof Combining (3.4) with (3.5) and taking the inner product with $\Delta^2\tilde{\phi}^{k+1}$ leads to

$$\begin{aligned} &\frac{1}{2\Delta t}(\|\Delta\tilde{\phi}^{k+1}\|^2 - \|\Delta\tilde{\phi}^{k}\|^2 + \|\Delta\tilde{\phi}^{k+1} - \Delta\tilde{\phi}^{k}\|^2) + M\lambda\|\Delta^2\tilde{\phi}^{k+1}\|^2 + M\lambda\gamma\|\nabla\Delta\tilde{\phi}^{k+1}\|^2 \\ &\quad = M\lambda(\Delta F'(\phi^k), \Delta^2\tilde{\phi}^{k+1}) - (\mathbf{u}^k \cdot \nabla\phi^k, \Delta^2\tilde{\phi}^{k+1}). \end{aligned} \tag{4.15}$$

The first term on the right hand side of (4.15) can be controlled by the following equation with the aid of (4.14):

$$\begin{aligned} M\lambda(\Delta F'(\phi^k), \Delta^2\tilde{\phi}^{k+1}) \le & \frac{M\lambda}{4}\|\Delta^2\tilde{\phi}^{k+1}\|^2 + C(K_1)\|\Delta F'(\phi^k)\|^2 \\ \le & \frac{M\lambda}{4}\|\Delta^2\tilde{\phi}^{k+1}\|^2 + \frac{M\lambda}{2}\|\Delta^2\tilde{\phi}^{k}\|^2 + C(K_1). \end{aligned} \tag{4.16}$$

Recalling (4.12), we have $\|\mathbf{u}^n\|_{H^1} \le C$. Then using (4.14) and lemmas 2.1 and 2.2, the last term on the right hand side of (4.15) can be bounded by

$$\begin{aligned} -(\mathbf{u}^k \cdot \nabla\phi^k, \Delta^2\tilde{\phi}^{k+1}) \le & \|\mathbf{u}^k\|_{L^6}\|\nabla\phi^k\|_{L^3}\|\Delta^2\tilde{\phi}^{k+1}\| \\ \le & C\|\mathbf{u}^k\|_{H^1}\|\nabla\phi^k\|^{1/2}\|\nabla\tilde{\phi}^k\|_{H^1}^{1/2}\|\Delta^2\tilde{\phi}^{k+1}\| \\ \le & C\|\mathbf{u}^k\|_{H^1}^2\|\nabla\tilde{\phi}^k\|_{H^1}^2 + \frac{M\lambda}{16}\|\Delta^2\tilde{\phi}^{k+1}\|^2 + C\|\mathbf{u}^k\|_{H^1}^2\|\nabla\phi^k\|^2 \\ \le & \frac{M\lambda}{4}\|\Delta^2\tilde{\phi}^{k+1}\|^2 + C\|\mathbf{u}^k\|_{H^1}^2(\|\Delta\tilde{\phi}^k\|^2 + C(K_1)). \end{aligned} \tag{4.17}$$

Combining (4.15) with (4.16)–(4.17) leads to

$$\begin{aligned} &\frac{1}{2\Delta t}(\|\Delta\tilde{\phi}^{k+1}\|^2 - \|\Delta\tilde{\phi}^{k}\|^2 + \|\Delta\tilde{\phi}^{k+1} - \Delta\tilde{\phi}^{k}\|^2) + \frac{M\lambda}{2}\|\Delta^2\tilde{\phi}^{k+1}\|^2 + M\lambda\gamma\|\nabla\Delta\tilde{\phi}^{k+1}\|^2 \\ &\quad \le \frac{M\lambda}{2}\|\Delta^2\tilde{\phi}^{k}\|^2 + C\|\mathbf{u}^k\|_{H^1}^2(\|\Delta\tilde{\phi}^k\|^2 + C(K_1)) + C(K_1). \end{aligned} \tag{4.18}$$

Then multiplying (4.18) by $2\Delta t$ and summing over k, $k = 0, 1, 2, \ldots, n$, we have

$$\begin{aligned}
&\|\Delta\tilde{\phi}^{n+1}\|^2 + M\lambda\Delta t\|\Delta^2\tilde{\phi}^{n+1}\|^2 + M\lambda\gamma\Delta t\sum_{k=0}^{n}\|\nabla\Delta\tilde{\phi}^{k+1}\|^2 \\
&\quad\le \|\Delta\tilde{\phi}^0\|^2 + M\lambda\Delta t\|\Delta^2\tilde{\phi}^0\|^2 + C\Delta t\sum_{k=0}^{n}\|\mathbf{u}^k\|_{H^1}^2\|\Delta\tilde{\phi}^k\|^2 + C(K_1),
\end{aligned} \tag{4.19}$$

which, together with lemma 2.3 and equations (3.5), (4.12) and (4.14), lead to the desired result. □

Lemma 4.3 *Under the assumption of Theorem 4.1, we have*

$$\begin{aligned}
&\lambda(\|\nabla\tilde{e}_\phi^{n+1}\|^2 - \|\nabla\tilde{e}_\phi^n\|^2 + \|\nabla\tilde{e}_\phi^{n+1} - \nabla\tilde{e}_\phi^n\|^2) + M\Delta t\|\nabla\tilde{e}_\mu^{n+1}\|^2 \\
&\quad + \lambda\gamma(\|\tilde{e}_\phi^{n+1}\|^2 - \|\tilde{e}_\phi^n\|^2 + \|\tilde{e}_\phi^{n+1} - \tilde{e}_\phi^n\|^2) + M\Delta t\|\tilde{e}_\mu^{n+1}\|^2 \\
&\quad \le C\Delta t\|\nabla\tilde{e}_\phi^{n+1}\|^2 + C\Delta t\|\tilde{e}_\phi^n\|^2 + C\Delta t\|\nabla\tilde{e}_\phi^n\|^2 + C\Delta t\|\nabla\tilde{e}_{\boldsymbol{u}}^n\|^2 \\
&\quad\quad + C(\|\nabla\tilde{\boldsymbol{u}}^n\|^2 + \|\tilde{\phi}^n\|_{H^1}^2)C_0^4(\Delta t)^5 + C(\Delta t)^3,
\end{aligned} \tag{4.20}$$

where C is a positive constant independent of Δt and C_0.

Proof Let R_ϕ^{n+1} be the truncation error defined by

$$R_\phi^{n+1} = \frac{\partial\phi(t^{n+1})}{\partial t} - \frac{\phi(t^{n+1}) - \phi(t^n)}{\Delta t} = \frac{1}{\Delta t}\int_{t^n}^{t^{n+1}}(t^n - t)\frac{\partial^2\phi}{\partial t^2}dt. \tag{4.21}$$

Subtracting (3.3a) at t^{n+1} from (3.4), we have

$$\frac{\tilde{e}_\phi^{n+1} - \tilde{e}_\phi^n}{\Delta t} - M\Delta\tilde{e}_\mu^{n+1} = (\mathbf{u}(t^{n+1})\cdot\nabla)\phi(t^{n+1}) - (\mathbf{u}^n\cdot\nabla)\phi^n + R_\phi^{n+1}. \tag{4.22}$$

Taking the inner product of (4.22) with $\tilde{e}_\mu^{n+1}$ and $\lambda\tilde{e}_\phi^{n+1}$, respectively leads to

$$\begin{aligned}
&\left(\frac{\tilde{e}_\phi^{n+1} - \tilde{e}_\phi^n}{\Delta t}, \tilde{e}_\mu^{n+1}\right) + M\|\nabla\tilde{e}_\mu^{n+1}\|^2 \\
&\quad = ((\mathbf{u}(t^{n+1})\cdot\nabla)\phi(t^{n+1}) - (\mathbf{u}^n\cdot\nabla)\phi^n, \tilde{e}_\mu^{n+1}) + (R_\phi^{n+1}, \tilde{e}_\mu^{n+1}),
\end{aligned} \tag{4.23}$$

and

$$\begin{aligned}
&\frac{\lambda}{2\Delta t}(\|\tilde{e}_\phi^{n+1}\|^2 - \|\tilde{e}_\phi^n\|^2 + \|\tilde{e}_\phi^{n+1} - \tilde{e}_\phi^n\|^2) \\
&\quad = \lambda((\mathbf{u}(t^{n+1})\cdot\nabla)\phi(t^{n+1}) - (\mathbf{u}^n\cdot\nabla)\phi^n, \tilde{e}_\phi^{n+1}) \\
&\quad\quad + \lambda(R_\phi^{n+1}, \tilde{e}_\phi^{n+1}) - M\lambda(\nabla\tilde{e}_\mu^{n+1}, \nabla\tilde{e}_\phi^{n+1}).
\end{aligned} \tag{4.24}$$

Subtracting (3.3b) at t^{n+1} from (3.5), we have

$$\tilde{e}_{\mu}^{n+1} = -\lambda\Delta\tilde{e}_{\phi}^{n+1} + \lambda\gamma\tilde{e}_{\phi}^{n+1} + \lambda F'(\phi^n) - \lambda F'(\phi(t^{n+1})). \tag{4.25}$$

Taking the inner product of (4.25) with $M\tilde{e}_{\mu}^{n+1}$ and $\frac{\tilde{e}_{\phi}^{n+1}-\tilde{e}_{\phi}^{n}}{\Delta t}$, respectively results in

$$\begin{aligned} M\|\tilde{e}_{\mu}^{n+1}\|^2 =& M\lambda(\nabla\tilde{e}_{\mu}^{n+1}, \nabla\tilde{e}_{\phi}^{n+1}) + M\lambda\gamma(\tilde{e}_{\phi}^{n+1}, \tilde{e}_{\mu}^{n+1}) \\ &+ M\lambda\left(F'(\phi^n) - F'(\phi(t^{n+1})), \tilde{e}_{\mu}^{n+1}\right), \end{aligned} \tag{4.26}$$

$$\begin{aligned} \left(\frac{\tilde{e}_{\phi}^{n+1}-\tilde{e}_{\phi}^{n}}{\Delta t}, \tilde{e}_{\mu}^{n+1}\right) =& \frac{\lambda}{2\Delta t}(\|\nabla\tilde{e}_{\phi}^{n+1}\|^2 - \|\nabla\tilde{e}_{\phi}^{n}\|^2 + \|\nabla\tilde{e}_{\phi}^{n+1} - \nabla\tilde{e}_{\phi}^{n}\|^2) \\ &+ \lambda\left(F'(\phi^n) - F'(\phi(t^{n+1})), \frac{\tilde{e}_{\phi}^{n+1}-\tilde{e}_{\phi}^{n}}{\Delta t}\right) \\ &+ \frac{\lambda\gamma}{2\Delta t}(\|\tilde{e}_{\phi}^{n+1}\|^2 - \|\tilde{e}_{\phi}^{n}\|^2 + \|\tilde{e}_{\phi}^{n+1} - \tilde{e}_{\phi}^{n}\|^2). \end{aligned} \tag{4.27}$$

Combining (4.23) with (4.24)–(4.27), we have

$$\begin{aligned} &\frac{\lambda}{2\Delta t}(\|\nabla\tilde{e}_{\phi}^{n+1}\|^2 - \|\nabla\tilde{e}_{\phi}^{n}\|^2 + \|\nabla\tilde{e}_{\phi}^{n+1} - \nabla\tilde{e}_{\phi}^{n}\|^2) + M\|\nabla\tilde{e}_{\mu}^{n+1}\|^2 \\ &+ \frac{\lambda\gamma}{\Delta t}(\|\tilde{e}_{\phi}^{n+1}\|^2 - \|\tilde{e}_{\phi}^{n}\|^2 + \|\tilde{e}_{\phi}^{n+1} - \tilde{e}_{\phi}^{n}\|^2) + M\|\tilde{e}_{\mu}^{n+1}\|^2 \\ &= ((\mathbf{u}(t^{n+1})\cdot\nabla)\phi(t^{n+1}) - (\mathbf{u}^n\cdot\nabla)\phi^n, \tilde{e}_{\mu}^{n+1} + \lambda\tilde{e}_{\phi}^{n+1}) + (R_{\phi}^{n+1}, \tilde{e}_{\mu}^{n+1}) \\ &\quad + \lambda(R_{\phi}^{n+1}, \tilde{e}_{\phi}^{n+1}) + M\lambda\gamma(\tilde{e}_{\phi}^{n+1}, \tilde{e}_{\mu}^{n+1}) \\ &\quad + \lambda\left(F'(\phi^n) - F'(\phi(t^{n+1}), M\tilde{e}_{\mu}^{n+1} - \frac{\tilde{e}_{\phi}^{n+1}-\tilde{e}_{\phi}^{n}}{\Delta t}\right). \end{aligned} \tag{4.28}$$

Using lemmas 2.1 and 2.2, the first term on the right hand side of (4.28) can be recast as

$$\begin{aligned} &\left((\mathbf{u}(t^{n+1})\cdot\nabla)\phi(t^{n+1}) - (\mathbf{u}^n\cdot\nabla)\phi^n, \tilde{e}_{\mu}^{n+1} + \lambda\tilde{e}_{\phi}^{n+1}\right) \\ &= \left((\mathbf{u}(t^{n+1})\cdot\nabla)\phi(t^{n+1}) - (\mathbf{u}(t^n)\cdot\nabla)\phi(t^n), \tilde{e}_{\mu}^{n+1} + \lambda\tilde{e}_{\phi}^{n+1}\right) \\ &\quad - \left((e_{\mathbf{u}}^n\cdot\nabla)\phi(t^n), \tilde{e}_{\mu}^{n+1}\right) - \left((\mathbf{u}^n\cdot\nabla)e_{\phi}^n, \tilde{e}_{\mu}^{n+1} + \lambda\tilde{e}_{\phi}^{n+1}\right) \\ &\le \frac{M}{8}\|\nabla\tilde{e}_{\mu}^{n+1}\|^2 + C\|\nabla\tilde{e}_{\phi}^{n+1}\|^2 + C\|\phi\|_{L^\infty(0,T;H^2(\Omega))}^2\|e_{\mathbf{u}}^n\|^2 \\ &\quad + C\|\mathbf{u}^n\|_{L^3}^2\|\nabla e_{\phi}^n\|_{L^2}^2 + C\|\mathbf{u}\|_{L^\infty(0,T;H^1(\Omega))}^2\|\phi\|_{W^{1,\infty}(0,T;H^1(\Omega))}^2(\Delta t)^2 \\ &\quad + C\|\mathbf{u}\|_{W^{1,\infty}(0,T;L^2(\Omega))}^2\|\phi\|_{L^\infty(0,T;H^1(\Omega))}^2(\Delta t)^2. \end{aligned} \tag{4.29}$$

Recalling lemma 4.2 and (4.22), the last term on the right hand side of (4.28) can be estimated by

$$
\begin{aligned}
&\lambda\left(F'(\phi^n)-F'(\phi(t^{n+1}), M\tilde{e}_\mu^{n+1}-\frac{\tilde{e}_\phi^{n+1}-\tilde{e}_\phi^n}{\Delta t}\right)\\
&=\lambda\left(F'(\phi^n)-F'(\phi(t^{n+1}), M\tilde{e}_\mu^{n+1}-R_\phi^{n+1}\right)\\
&\quad-\lambda\left(F'(\phi^n)-F'(\phi(t^{n+1}), (\mathbf{u}(t^{n+1})\cdot\nabla)\phi(t^{n+1})-(\mathbf{u}^n\cdot\nabla)\phi^n\right)\\
&\quad+\lambda\left(\nabla(F'(\phi^n)-F'(\phi(t^{n+1})), M\nabla\tilde{e}_\mu^{n+1}\right)\\
&\leq\frac{M}{8}\|\tilde{e}_\mu^{n+1}\|^2+\frac{M}{8}\|\nabla\tilde{e}_\mu^{n+1}\|^2+C\|e_\phi^n\|^2+C\|\nabla e_\phi^n\|^2\\
&\quad+C\|\phi\|^2_{W^{1,\infty}(0,T;H^1(\Omega))}(\Delta t)^2+C\|\phi\|^2_{L^\infty(0,T;H^2(\Omega))}\|e_{\mathbf{u}}^n\|^2\\
&\quad+C\|\mathbf{u}^n\|^2_{L^3}\|\nabla e_\phi^n\|^2_{L^2}+C\|\mathbf{u}\|^2_{L^\infty(0,T;H^1(\Omega))}\|\phi\|^2_{W^{1,\infty}(0,T;H^1(\Omega))}(\Delta t)^2\\
&\quad+C\|\mathbf{u}\|^2_{W^{1,\infty}(0,T;L^2(\Omega))}\|\phi\|^2_{L^\infty(0,T;H^1(\Omega))}(\Delta t)^2\\
&\quad+C\|\phi\|^2_{W^{2,\infty}(0,T;L^2(\Omega))}(\Delta t)^2.
\end{aligned}
\tag{4.30}
$$

Using Poincaré inequality, we have $\|e_{\mathbf{u}}^n\|^2\leq C\|\nabla e_{\mathbf{u}}^n\|^2$ and using (3.8) and (4.11), we have

$$
\begin{aligned}
\|\nabla e_{\mathbf{u}}^n\|^2 &\leq 2\|\nabla\tilde{e}_{\mathbf{u}}^n\|^2+2|1-\eta^n|^2\|\nabla\tilde{\mathbf{u}}^n\|^2\\
&\leq 2\|\nabla\tilde{e}_{\mathbf{u}}^n\|^2+2\|\nabla\tilde{\mathbf{u}}^n\|^2C_0^4(\Delta t)^4,
\end{aligned}
\tag{4.31}
$$

$$
\|e_\phi^n\|^2_{H^2}\leq 2\|\tilde{e}_\phi^n\|^2_{H^2}+2\|\tilde{\phi}^n\|^2_{H^2}C_0^4(\Delta t)^4. \tag{4.32}
$$

Then combining (4.28) with (4.29) and (4.30) and multiplying $2\Delta t$ on both sides, we have

$$
\begin{aligned}
&\lambda(\|\nabla\tilde{e}_\phi^{n+1}\|^2-\|\nabla\tilde{e}_\phi^n\|^2+\|\nabla\tilde{e}_\phi^{n+1}-\nabla\tilde{e}_\phi^n\|^2)+M\Delta t\|\nabla\tilde{e}_\mu^{n+1}\|^2\\
&\quad+\lambda\gamma(\|\tilde{e}_\phi^{n+1}\|^2-\|\tilde{e}_\phi^n\|^2+\|\tilde{e}_\phi^{n+1}-\tilde{e}_\phi^n\|^2)+M\Delta t\|\tilde{e}_\mu^{n+1}\|^2\\
&\leq C\Delta t\|\nabla\tilde{e}_\phi^{n+1}\|^2+C\Delta t\|\tilde{e}_\phi^n\|^2+C\Delta t\|\nabla\tilde{e}_\phi^n\|^2+C\Delta t\|\mathbf{u}^n\|^2_{L^3}\|\nabla\tilde{e}_\phi^n\|^2\\
&\quad+C\|\phi\|^2_{W^{1,\infty}(0,T;H^1(\Omega))}(\Delta t)^3+C\Delta t\|\phi\|^2_{L^\infty(0,T;H^2(\Omega))}\|\nabla\tilde{e}_{\mathbf{u}}^n\|^2\\
&\quad+C\|\mathbf{u}\|^2_{L^\infty(0,T;H^1(\Omega))}\|\phi\|^2_{W^{1,\infty}(0,T;H^1(\Omega))}(\Delta t)^3+C\|\nabla\tilde{\mathbf{u}}^n\|^2C_0^4(\Delta t)^5\\
&\quad+C\|\mathbf{u}\|^2_{W^{1,\infty}(0,T;L^2(\Omega))}\|\phi\|^2_{L^\infty(0,T;H^1(\Omega))}(\Delta t)^3+C\|\tilde{\phi}^n\|^2_{H^1}C_0^4(\Delta t)^5\\
&\quad+C\|\phi\|^2_{W^{2,\infty}(0,T;L^2(\Omega))}(\Delta t)^3,
\end{aligned}
\tag{4.33}
$$

which implies the desired result (4.20). □

Step 2: Estimates for H^2 bounds of $\tilde{e}_{\mathbf{u}}^{n+1}$. We first establish error estimate for the commutator of the Laplacian and Leray-Helmholtz projection operators to bound the part of pressure. Similar to [21], we let $\mathcal{P}$ denote the Leray-Helmholtz projection operator onto divergence-free fields, defined as follows. Given any $\mathbf{b}\in L^2(\Omega,\mathbb{R}^d)$, there is a unique $q\in H^1(\Omega)$ with $\int_\Omega q=0$ such that $\mathcal{P}\mathbf{b}=\mathbf{b}+\nabla q$ satisfies

$$
(\mathbf{b}+\nabla q,\nabla\phi)=(\mathcal{P}\mathbf{b},\nabla\phi)=0,\ \forall\phi\in H^1(\Omega). \tag{4.34}
$$

Then for $\mathbf{u}\in L^2(\Omega,\mathbb{R}^d)$, we have [21]

$$
\Delta\mathcal{P}\mathbf{u}=\Delta\mathbf{u}-\nabla\nabla\cdot\mathbf{u}=-\nabla\times\nabla\times\mathbf{u}. \tag{4.35}
$$

Next we recall the estimate for commutator of the Laplacian and Leray-Helmholtz projection operators.

Lemma 4.4 [21] *Let $\Omega \subset \mathbb{R}^d$ be a connected bounded domain with C^3 boundary. Then for any $\epsilon > 0$, there exists a positive constant $C \geq 0$ such that for all vector fields $\boldsymbol{u} \in H^2 \cap H_0^1(\Omega, \mathbb{R}^d)$,*

$$\int_\Omega |(\Delta\mathcal{P} - \mathcal{P}\Delta)\boldsymbol{u}|^2 \leq \left(\frac{1}{2} + \epsilon\right) \int_\Omega |\Delta \boldsymbol{u}|^2 + C \int_\Omega |\nabla \boldsymbol{u}|^2. \tag{4.36}$$

We define the Stokes pressure $p_s(\mathbf{u})$ by

$$\nabla p_s(\mathbf{u}) = (\Delta\mathcal{P} - \mathcal{P}\Delta)\mathbf{u}, \tag{4.37}$$

where the Stokes pressure is generated by the tangential part of vorticity at the boundary in two and three dimensions by

$$\int_\Omega \nabla p_s(\mathbf{u}) \cdot \nabla\phi = \int_\Gamma (\nabla \times \mathbf{u}) \cdot (\mathbf{n} \times \nabla\phi), \ \forall \phi \in H^1(\Omega). \tag{4.38}$$

Then by using (4.35), we have

$$\nabla p_s(\mathbf{u}) = (\Delta\mathcal{P} - \mathcal{P}\Delta)\mathbf{u} = (I - \mathcal{P})\Delta\mathbf{u} - \nabla\nabla \cdot \mathbf{u} = (I - \mathcal{P})(\Delta\mathbf{u} - \nabla\nabla \cdot \mathbf{u}). \tag{4.39}$$

Recalling (4.34), we have

$$\int_\Omega \nabla p_s(\mathbf{u}) \cdot \nabla\phi = \int_\Omega (\Delta\mathbf{u} - \nabla\nabla \cdot \mathbf{u}) \cdot \nabla\phi, \ \forall \phi \in H^1(\Omega). \tag{4.40}$$

Lemma 4.5 *Under the assumption of Theorem 4.1, we have*

$$\begin{aligned}
&\frac{M}{2K_3}(\|\nabla\tilde{e}_{\boldsymbol{u}}^{n+1}\|^2 - \|\nabla\tilde{e}_{\boldsymbol{u}}^{n}\|^2 + \|\nabla\tilde{e}_{\boldsymbol{u}}^{n+1} - \nabla\tilde{e}_{\boldsymbol{u}}^{n}\|^2) + \frac{M}{2K_3}\left(1 - \frac{3(1-\alpha)}{8}\right)\nu\Delta t\|\Delta\tilde{e}_{\boldsymbol{u}}^{n+1}\|^2 \\
&\leq \frac{M}{2K_3}\left(\alpha + \frac{(1-\alpha)}{8}\right)\nu\Delta t\|\Delta\tilde{e}_{\boldsymbol{u}}^{n}\|^2 + C\Delta t\|\nabla\tilde{e}_{\boldsymbol{u}}^{n}\|^2 \\
&\quad + C\Delta t\|\tilde{e}_{\phi}^{n}\|^2 + C\Delta t\|\nabla\tilde{e}_{\phi}^{n}\|^2 + \frac{M}{2}\Delta t(\|\tilde{e}_{\mu}^{n}\|^2 + \|\nabla\tilde{e}_{\mu}^{n}\|^2) \\
&\quad + C(\|\Delta\tilde{\boldsymbol{u}}^n\|^2 + \|\nabla\tilde{\boldsymbol{u}}^n\|^2)C_0^4(\Delta t)^5 \\
&\quad + C\|\tilde{\phi}^n\|_{H^2}^2 C_0^4(\Delta t)^5 + C\|\nabla\tilde{\mu}^n\|^2 C_0^4(\Delta t)^5 + C(\Delta t)^3,
\end{aligned} \tag{4.41}$$

where the positive constant C is independent of Δt and C_0 and the positive constant α satisfies $\frac{1}{2} < \alpha < 1$.

Proof Let $\mathbf{R}_{\mathbf{u}}^{n+1}$ be the truncation error defined by

$$\mathbf{R}_{\mathbf{u}}^{n+1} = \frac{\partial\mathbf{u}(t^{n+1})}{\partial t} - \frac{\mathbf{u}(t^{n+1}) - \mathbf{u}(t^n)}{\Delta t} = \frac{1}{\Delta t}\int_{t^n}^{t^{n+1}} (t^n - t)\frac{\partial^2\mathbf{u}}{\partial t^2}dt. \tag{4.42}$$

Subtracting (3.3c) at t^{n+1} from (3.6), we obtain

$$\begin{aligned}
&\frac{\tilde{e}_{\mathbf{u}}^{n+1} - \tilde{e}_{\mathbf{u}}^{n}}{\Delta t} - \nu\Delta\tilde{e}_{\mathbf{u}}^{n+1} = (\mathbf{u}(t^{n+1}) \cdot \nabla)\mathbf{u}(t^{n+1}) - \mathbf{u}^n \cdot \nabla\mathbf{u}^n \\
&\quad - \nabla(p^n - p(t^{n+1})) + \mu^n\nabla\phi^n - \mu(t^{n+1})\nabla\phi(t^{n+1}) + \mathbf{R}_{\mathbf{u}}^{n+1}.
\end{aligned} \tag{4.43}$$

Next we establish an error equation for pressure corresponding to (3.11) by

$$
\begin{aligned}
(\nabla e_p^{n+1}, \nabla q) = &\big((\mathbf{u}(t^{n+1}) \cdot \nabla)\mathbf{u}(t^{n+1}) - (\mathbf{u}^{n+1} \cdot \nabla)\mathbf{u}^{n+1}, \nabla q\big) \\
&+ \big(\mu^{n+1}\nabla\phi^{n+1} - \mu(t^{n+1})\nabla\phi(t^{n+1}), \nabla q\big) \\
&- (\nu\nabla \times \nabla \times \tilde{e}_{\mathbf{u}}^{n+1}, \nabla q), \ \forall q \in H^1(\Omega).
\end{aligned}
\tag{4.44}
$$

Taking $q = e_p^{n+1}$ in (4.44) leads to

$$
\begin{aligned}
\|\nabla e_p^{n+1}\| \le & \nu\|\nabla p_s^{n+1}(\tilde{e}_{\mathbf{u}})\| + \|e_{\mathbf{u}}^{n+1}\|_{L^6}\|\nabla\mathbf{u}(t^{n+1})\|_{L^3} + \|\mathbf{u}^{n+1}\|_{L^6}\|\nabla e_{\mathbf{u}}^{n+1}\|_{L^3} \\
&+ \|e_\mu^{n+1}\|_{L^6}\|\nabla\phi^{n+1}\|_{L^3} + \|\mu(t^{n+1})\|_{L^6}\|\nabla e_\phi^{n+1}\|_{L^3}.
\end{aligned}
\tag{4.45}
$$

Recalling (4.40) and lemma 4.4, we have

$$
\nu\|\nabla p_s^{n+1}(\tilde{e}_{\mathbf{u}})\|^2 \le \nu\alpha\|\Delta\tilde{e}_{\mathbf{u}}^{n+1}\|^2 + \nu C_\alpha\|\nabla\tilde{e}_{\mathbf{u}}^{n+1}\|^2, \tag{4.46}
$$

where the positive constant $\frac{1}{2} < \alpha < 1$.

Taking the inner product of (4.43) with $-2\Delta t\Delta\tilde{e}_{\mathbf{u}}^{n+1}$ and using lemmas 2.1 and 2.2, we have

$$
\begin{aligned}
&(\|\nabla\tilde{e}_{\mathbf{u}}^{n+1}\|^2 - \|\nabla\tilde{e}_{\mathbf{u}}^{n}\|^2 + \|\nabla\tilde{e}_{\mathbf{u}}^{n+1} - \nabla\tilde{e}_{\mathbf{u}}^{n}\|^2) + 2\nu\Delta t\|\Delta\tilde{e}_{\mathbf{u}}^{n+1}\|^2 \\
&\quad \le 2\nu\Delta t\|\Delta\tilde{e}_{\mathbf{u}}^{n+1}\|\|\nabla p_s^n(\tilde{e}_{\mathbf{u}})\| + 4\Delta t\|\Delta\tilde{e}_{\mathbf{u}}^{n+1}\|(\|e_{\mathbf{u}}^n\|_{L^6}\|\nabla\mathbf{u}(t^n)\|_{L^3} + \|\mathbf{u}^n\|_{L^6}\|\nabla e_{\mathbf{u}}^n\|_{L^3}) \\
&\quad + 4\Delta t\|\Delta\tilde{e}_{\mathbf{u}}^{n+1}\|(\|e_\mu^n\|_{L^6}\|\nabla\phi^n\|_{L^3} + \|\mu(t^n)\|_{L^6}\|\nabla e_\phi^n\|_{L^3}) \\
&\quad + 2\Delta t\|\Delta\tilde{e}_{\mathbf{u}}^{n+1}\|(\|p(t^n) - p(t^{n+1})\| + \|\mathbf{R}_{\mathbf{u}}^{n+1}\| + \|(\mathbf{u}(t^{n+1}) \cdot \nabla)\mathbf{u}(t^{n+1}) - (\mathbf{u}(t^n) \cdot \nabla)\mathbf{u}(t^n)\|) \\
&\quad + 2\Delta t\|\Delta\tilde{e}_{\mathbf{u}}^{n+1}\|\|\mu(t^{n+1})\nabla\phi(t^{n+1}) - \mu(t^n)\nabla\phi(t^n)\|.
\end{aligned}
\tag{4.47}
$$

Using Cauchy-Schwarz inequality, the first term on the right hand side of (4.47) can be estimated by

$$
2\nu\Delta t\|\Delta\tilde{e}_{\mathbf{u}}^{n+1}\|\|\nabla p_s^n(\tilde{e}_{\mathbf{u}})\| \le \nu\Delta t\|\Delta\tilde{e}_{\mathbf{u}}^{n+1}\|^2 + \nu\alpha\Delta t\|\Delta\tilde{e}_{\mathbf{u}}^{n}\|^2 + \nu C_\alpha\Delta t\|\nabla\tilde{e}_{\mathbf{u}}^{n}\|^2. \tag{4.48}
$$

Recalling (4.12), we can obtain $\|\mathbf{u}^n\|_{H^2} \le C$ and by using (3.8) and (4.11), we have

$$
\|\Delta e_{\mathbf{u}}^n\|^2 \le 2\|\Delta\tilde{e}_{\mathbf{u}}^n\|^2 + 2|1 - \eta^n|^2\|\Delta\tilde{\mathbf{u}}^n\|^2 \le 2\|\Delta\tilde{e}_{\mathbf{u}}^n\|^2 + 2\|\Delta\tilde{\mathbf{u}}^n\|^2 C_0^4(\Delta t)^4. \tag{4.49}
$$

Thus the second term on the right hand side of (4.47) can be estimated by

$$
\begin{aligned}
&4\Delta t\|\Delta\tilde{e}_{\mathbf{u}}^{n+1}\|(\|e_{\mathbf{u}}^n\|_{L^6}\|\nabla\mathbf{u}(t^n)\|_{L^3} + \|\mathbf{u}^n\|_{L^6}\|\nabla e_{\mathbf{u}}^n\|_{L^3}) \\
&\le \frac{(1-\alpha)\nu}{8}\Delta t\|\Delta\tilde{e}_{\mathbf{u}}^{n+1}\|^2 + C\Delta t\|\nabla e_{\mathbf{u}}^n\|^2\|\nabla\mathbf{u}(t^n)\|\|\nabla\mathbf{u}(t^n)\|_{H^1} \\
&\quad + C\Delta t\|\nabla\mathbf{u}^n\|^2\|\nabla e_{\mathbf{u}}^n\|\|\nabla e_{\mathbf{u}}^n\|_{H^1} \\
&\le \frac{(1-\alpha)\nu}{8}\Delta t\|\Delta\tilde{e}_{\mathbf{u}}^{n+1}\|^2 + \frac{(1-\alpha)\nu}{16}\Delta t\|\Delta e_{\mathbf{u}}^n\|^2 + C\Delta t\|\nabla e_{\mathbf{u}}^n\|^2 \\
&\le \frac{(1-\alpha)\nu}{8}\Delta t\|\Delta\tilde{e}_{\mathbf{u}}^{n+1}\|^2 + \frac{(1-\alpha)\nu}{8}\Delta t\|\Delta\tilde{e}_{\mathbf{u}}^n\|^2 + C\Delta t\|\nabla\tilde{e}_{\mathbf{u}}^n\|^2 \\
&\quad + C(\|\Delta\tilde{\mathbf{u}}^n\|^2 + \|\nabla\tilde{\mathbf{u}}^n\|^2)C_0^4(\Delta t)^5.
\end{aligned}
\tag{4.50}
$$

Using (4.25), we have

$$
\Delta\tilde{e}_\phi^{n+1} = \gamma\tilde{e}_\phi^{n+1} + F'(\phi^n) - F'(\phi(t^{n+1})) - \frac{1}{\lambda}\tilde{e}_\mu^{n+1}. \tag{4.51}
$$

Recalling (4.12) leads to $\|\mu^n\|_{H^1} \le C$. In addition, we have

$$\|e_\phi^n\|_{H^2}^2 \le 2\|\tilde{e}_\phi^n\|_{H^2}^2 + 2\|\tilde{\phi}^n\|_{H^2}^2 C_0^4(\Delta t)^4, \tag{4.52}$$

$$\|\nabla e_\mu^n\|^2 \le 2\|\nabla \tilde{e}_\mu^n\|^2 + 2\|\nabla \tilde{\mu}^n\|^2 C_0^4(\Delta t)^4. \tag{4.53}$$

Hence the third term on the right hand side of (4.47) can be bounded by

$$\begin{aligned}
&4\Delta t\|\Delta \tilde{e}_{\mathbf{u}}^{n+1}\|(\|e_\mu^n\|_{L^6}\|\nabla\phi^n\|_{L^3} + \|\mu(t^n)\|_{L^6}\|\nabla e_\phi^n\|_{L^3})\\
&\le 4\Delta t\|\Delta \tilde{e}_{\mathbf{u}}^{n+1}\|(\|e_\mu^n\|_{H^1}\|\nabla\phi^n\|^{1/2}\|\nabla\phi^n\|_{H^1}^{1/2} + \|\mu(t^n)\|_{H^1}\|\nabla e_\phi^n\|^{1/2}\|\nabla e_\phi^n\|_{H^1}^{1/2})\\
&\le \frac{(1-\alpha)\nu}{8}\Delta t\|\Delta \tilde{e}_{\mathbf{u}}^{n+1}\|^2 + K_3\Delta t(\|\tilde{e}_\mu^n\|^2 + \|\nabla \tilde{e}_\mu^n\|^2) + C\Delta t\|\tilde{e}_\phi^n\|^2\\
&\quad + C\Delta t\|\nabla \tilde{e}_\phi^n\|^2 + C\|\tilde{\phi}^n\|_{H^2}^2 C_0^4(\Delta t)^5 + C\|\nabla\tilde{\mu}^n\|^2 C_0^4(\Delta t)^5\\
&\quad + C\|\phi\|_{W^{1,\infty}(0,T;H^1(\Omega))}^2\|\mu\|_{L^\infty(0,T;H^1(\Omega))}^2(\Delta t)^3,
\end{aligned} \tag{4.54}$$

where K_3 is a positive constant which is independent of Δt and C_0.

Using Cauchy-Schwarz inequality, the last two terms on the right hand side of (4.47) can be bounded by

$$\begin{aligned}
&2\Delta t\|\Delta \tilde{e}_{\mathbf{u}}^{n+1}\|(\|p(t^n) - p(t^{n+1})\| + \|\mathbf{R}_{\mathbf{u}}^{n+1}\| + \|(\mathbf{u}(t^{n+1})\cdot\nabla)\mathbf{u}(t^{n+1}) - (\mathbf{u}(t^n)\cdot\nabla)\mathbf{u}(t^n)\|)\\
&\quad + 2\Delta t\|\Delta \tilde{e}_{\mathbf{u}}^{n+1}\|\|\mu(t^{n+1})\nabla\phi(t^{n+1}) - \mu(t^n)\nabla\phi(t^n)\|\\
&\le \frac{(1-\alpha)\nu}{8}\Delta t\|\Delta \tilde{e}_{\mathbf{u}}^{n+1}\|^2 + C(\Delta t)^2\left(\int_{t^n}^{t^{n+1}}\|p_t\|^2 dt + \int_{t^n}^{t^{n+1}}\|\mathbf{u}_{tt}\|^2 dt\right)\\
&\quad + C(\Delta t)^2\left(\int_{t^n}^{t^{n+1}}\|\nabla\mathbf{u}_t\|^2 dt\|\mathbf{u}(t^{n+1})\|_{H^2}^2 + \|\mathbf{u}(t^n)\|_{H^1}^2\int_{t^n}^{t^{n+1}}\|\mathbf{u}_t\|_{H^2}^2 dt\right)\\
&\quad + C(\Delta t)^2\left(\int_{t^n}^{t^{n+1}}\|\nabla\phi_t\|^2 dt\|\mu(t^{n+1})\|_{H^2}^2 + \|\phi(t^n)\|_{H^1}^2\int_{t^n}^{t^{n+1}}\|\mu_t\|_{H^2}^2 dt\right).
\end{aligned} \tag{4.55}$$

Finally, combining (4.47) with (4.48)–(4.55), we obtain

$$\begin{aligned}
&(\|\nabla \tilde{e}_{\mathbf{u}}^{n+1}\|^2 - \|\nabla \tilde{e}_{\mathbf{u}}^n\|^2 + \|\nabla \tilde{e}_{\mathbf{u}}^{n+1} - \nabla \tilde{e}_{\mathbf{u}}^n\|^2) + (1 - \frac{3(1-\alpha)}{8})\nu\Delta t\|\Delta \tilde{e}_{\mathbf{u}}^{n+1}\|^2\\
&\le \left(\alpha + \frac{(1-\alpha)}{8}\right)\nu\Delta t\|\Delta \tilde{e}_{\mathbf{u}}^n\|^2 + C\Delta t\|\nabla \tilde{e}_{\mathbf{u}}^n\|^2 + C\Delta t\|\tilde{e}_\phi^n\|^2 + C\Delta t\|\nabla \tilde{e}_\phi^n\|^2\\
&\quad + K_3\Delta t(\|\tilde{e}_\mu^n\|^2 + \|\nabla \tilde{e}_\mu^n\|^2)\\
&\quad + C(\|\Delta\tilde{\mathbf{u}}^n\|^2 + \|\nabla\tilde{\mathbf{u}}^n\|^2)C_0^4(\Delta t)^5\\
&\quad + C\|\tilde{\phi}^n\|_{H^2}^2 C_0^4(\Delta t)^5 + C\|\nabla\tilde{\mu}^n\|^2 C_0^4(\Delta t)^5\\
&\quad + C(|\mathbf{u}\|_{W^{1,\infty}(0,T;H^2(\Omega))}^2 + \|\phi\|_{W^{1,\infty}(0,T;H^1(\Omega))}^2 + \|\mu\|_{W^{1,\infty}(0,T;H^2(\Omega))}^2)(\Delta t)^3\\
&\quad + C(\|p\|_{W^{1,\infty}(0,T;L^2(\Omega))}^2 + \|\mathbf{u}\|_{W^{2,\infty}(0,T;L^2(\Omega))}^2)(\Delta t)^3,
\end{aligned} \tag{4.56}$$

which leads to the desired result (4.41) by multiplying $\frac{M}{2K_3}$ on both sides of (4.56). □

Combining lemmas 4.3 and 4.5, we have

$$\begin{aligned}
&\lambda(\|\nabla \tilde{e}_\phi^{k+1}\|^2 - \|\nabla \tilde{e}_\phi^k\|^2 + \|\nabla \tilde{e}_\phi^{k+1} - \nabla \tilde{e}_\phi^k\|^2) + \lambda\gamma(\|\tilde{e}_\phi^{k+1}\|^2 - \|\tilde{e}_\phi^k\|^2 + \|\tilde{e}_\phi^{k+1} - \tilde{e}_\phi^k\|^2)\\
&\quad + M\Delta t\|\nabla \tilde{e}_\mu^{k+1}\|^2 + M\Delta t\|\tilde{e}_\mu^{k+1}\|^2 + \frac{M}{2K_3}(\|\nabla \tilde{e}_{\mathbf{u}}^{k+1}\|^2 - \|\nabla \tilde{e}_{\mathbf{u}}^k\|^2 + \|\nabla \tilde{e}_{\mathbf{u}}^{k+1} - \nabla \tilde{e}_{\mathbf{u}}^k\|^2)
\end{aligned}$$

$$
\begin{aligned}
&+ \frac{M}{2K_3}\left(1-\frac{3(1-\alpha)}{8}\right)\nu\Delta t\|\Delta\tilde{e}_{\mathbf{u}}^{k+1}\|^2 \\
&\leq C_1\Delta t\|\nabla\tilde{e}_{\phi}^{k+1}\|^2 + C\Delta t\|\tilde{e}_{\phi}^{k}\|^2 + C\Delta t\|\nabla\tilde{e}_{\phi}^{k}\|^2 + C\left(\|\nabla\tilde{\mathbf{u}}^k\|^2+\|\tilde{\phi}^k\|_{H^1}^2\right)C_0^4(\Delta t)^5 \\
&+ C(\Delta t)^3 + \frac{M}{2K_3}(\alpha+\frac{(1-\alpha)}{8})\nu\Delta t\|\Delta\tilde{e}_{\mathbf{u}}^{k}\|^2 + C\Delta t\|\nabla\tilde{e}_{\mathbf{u}}^{k}\|^2 + C\Delta t\|\tilde{e}_{\phi}^{k}\|^2 + C\Delta t\|\nabla\tilde{e}_{\phi}^{k}\|^2 \\
&+ \frac{M}{2}\Delta t(\|\tilde{e}_{\mu}^{k}\|^2+\|\nabla\tilde{e}_{\mu}^{k}\|^2) + C(\|\Delta\tilde{\mathbf{u}}^k\|^2+\|\nabla\tilde{\mathbf{u}}^k\|^2)C_0^4(\Delta t)^5 \\
&+ C\|\tilde{\phi}^k\|_{H^2}^2C_0^4(\Delta t)^5 + C\|\nabla\tilde{\mu}^k\|^2C_0^4(\Delta t)^5 + C(\Delta t)^3. \qquad (4.57)
\end{aligned}
$$

Summing (4.57) over k, $k = 0, 1, 2, \ldots, n$, using boundedness estimates (4.12), (4.14), lemma 4.2 and applying the discrete Gronwall lemma 2.3 under the condition that $\Delta t \leq \frac{\lambda}{2C_1}$, we can arrive at

$$
\begin{aligned}
&\|\tilde{e}_{\phi}^{n+1}\|^2 + \|\nabla\tilde{e}_{\phi}^{n+1}\|^2 + \Delta t\sum_{k=0}^{n}\|\tilde{e}_{\mu}^{k+1}\|^2 + \Delta t\sum_{k=0}^{n}\|\nabla\tilde{e}_{\mu}^{k+1}\|^2 \\
&\quad + \|\nabla\tilde{e}_{\mathbf{u}}^{n+1}\|^2 + \Delta t\sum_{k=0}^{n}\|\Delta\tilde{e}_{\mathbf{u}}^{k+1}\|^2 \qquad (4.58)\\
&\quad \leq C_2\left(1+C_0^4(\Delta t)^2\right)(\Delta t)^2, \ \forall n \leq T/\Delta t,
\end{aligned}
$$

where C_2 is independent of C_0 and Δt.

Step 3: Estimates for $|1-\xi^{n+1}|$ with the new relaxation. We finish the induction process by establishing the estimates for $|1-\xi^{n+1}|$. Let S_r^{k+1} be the truncation error defined by

$$
S_r^{k+1} = \frac{\partial r(t^{k+1})}{\partial t} - \frac{R(t^{k+1})-R(t^k)}{\Delta t} = \frac{1}{\Delta t}\int_{t^k}^{t^{k+1}}(t^k-t)\frac{\partial^2 r}{\partial t^2}dt. \qquad (4.59)
$$

Subtracting (3.3e) at t^{k+1} from (3.7), we obtain

$$
\begin{aligned}
\frac{\tilde{e}_R^{k+1}-e_R^k}{\Delta t} =& -\frac{\tilde{R}^{k+1}}{E(\tilde{\phi}^{k+1},\tilde{\mathbf{u}}^{k+1})+\kappa_0}\left(M\|\nabla\tilde{\mu}^{k+1}\|^2+\nu\|\nabla\mathbf{u}^k\|^2\right) \\
&+ \frac{r(t^{k+1})}{E(\phi(t^{k+1}),\mathbf{u}(t^{k+1}))+\kappa_0}\left(M\|\nabla\mu(t^{k+1})\|^2+\nu\|\nabla\mathbf{u}(t^{k+1})\|^2\right) + S_r^{k+1}. \qquad (4.60)
\end{aligned}
$$

Thus we can obtain an error equation corresponding to (3.18)

$$
e_R^k = \sigma^k\tilde{e}_R^k + (1-\sigma^k)\left(E(\phi^k,\mathbf{u}^k) - E(\phi(t^k),\mathbf{u}(t^k))\right). \qquad (4.61)
$$

Plugging (4.61) into (4.60) leads to

$$
\begin{aligned}
\tilde{e}_R^{k+1} - \sigma^k\tilde{e}_R^k =& (1-\sigma^k)\left(E(\phi^k,\mathbf{u}^k) - E(\phi(t^k),\mathbf{u}(t^k))\right) \\
&- \Delta t\frac{\tilde{R}^{k+1}}{E(\tilde{\phi}^{k+1},\tilde{\mathbf{u}}^{k+1})+\kappa_0}\left(M\|\nabla\tilde{\mu}^{k+1}\|^2+\nu\|\nabla\mathbf{u}^k\|^2\right) \qquad (4.62)\\
&+ \Delta t\frac{r(t^{k+1})}{E(\phi(t^{k+1}),\mathbf{u}(t^{k+1}))+\kappa_0}\left(M\|\nabla\mu(t^{k+1})\|^2+\nu\|\nabla\mathbf{u}(t^{k+1})\|^2\right) + \Delta t S_r^{k+1}.
\end{aligned}
$$

Next we continue the error estimates in the following two cases $R^{k-1} \geq E(\phi^k,\mathbf{u}^k)+\kappa_0$ and $R^{k-1} < E(\phi^k,\mathbf{u}^k)+\kappa_0$.

Case I: Under the condition that $R^{k-1} \geq E(\phi^k, \mathbf{u}^k) + \kappa_0$, we have from Lemma 3.2 that $\sigma^k = 0$. Thus multiplying (4.62) with $\tilde{e}_R^{k+1}$ and using the triangle inequality, we have

$$\begin{aligned} |\tilde{e}_R^{k+1}|^2 = &\Big(E(\phi^k, \mathbf{u}^k) - E(\phi(t^k), \mathbf{u}(t^k)), \tilde{e}_R^{k+1}\Big) \\ &+ \Delta t(S_r^{k+1}, \tilde{e}_R^{k+1}) + H(\tilde{R}^{k+1}, \tilde{\mu}^{k+1}, \mathbf{u}^k), \end{aligned} \tag{4.63}$$

where

$$\begin{aligned} H(\tilde{R}^{k+1}, \tilde{\mu}^{k+1}, \mathbf{u}^k) =& \Delta t \Bigg(\frac{\tilde{R}^{k+1}}{E(\tilde{\phi}^{k+1}, \tilde{\mathbf{u}}^{k+1}) + \kappa_0} \Big(M\|\nabla \tilde{\mu}^{k+1}\|^2 + \nu\|\nabla \mathbf{u}^k\|^2\Big) \\ &- \Big(M\|\nabla \mu(t^{k+1})\|^2 + \nu\|\nabla \mathbf{u}(t^{k+1})\|^2\Big), \tilde{e}_R^{k+1}\Bigg). \end{aligned}$$

Using (4.14) and (4.58), we have

$$\begin{aligned} &|E(\phi^k, \mathbf{u}^k) - E(\phi(t^k), \mathbf{u}(t^k))| \\ &\quad \leq |E(\phi^k, \mathbf{u}^k) - E(\tilde{\phi}^k, \tilde{\mathbf{u}}^k)| + |E(\tilde{\phi}^k, \tilde{\mathbf{u}}^k) - E(\phi(t^k), \mathbf{u}(t^k))| \\ &\quad \leq CK_1(\|\nabla\phi^k - \nabla\tilde{\phi}^k\| + \|\phi^k - \tilde{\phi}^k\| + \|\mathbf{u}^k - \tilde{\mathbf{u}}^k\|) \\ &\qquad + C(\|\nabla\tilde{\phi}^k - \nabla\phi(t^k)\| + \|\tilde{\phi}^k - \phi(t^k)\| + \|\tilde{\mathbf{u}}^k - \mathbf{u}(t^k)\|) \\ &\qquad + \lambda \int_\Omega |F(\phi^k) - F(\phi(t^k))| d\mathbf{x} \\ &\quad \leq C|1 - \eta_1^k|(\|\tilde{\phi}^k\|_{H^1} + \|\tilde{\mathbf{u}}^k\|) + C\Delta t \\ &\quad \leq C(\|\tilde{\phi}^k\|_{H^1} + \|\tilde{\mathbf{u}}^k\|)C_0^2(\Delta t)^2 + C\Delta t. \end{aligned} \tag{4.64}$$

Then the first term on the right hand side of (4.63) can be estimated as

$$\begin{aligned} &\Big(E(\phi^k, \mathbf{u}^k) - E(\phi(t^k), \mathbf{u}(t^k)), \tilde{e}_R^{k+1}\Big) \\ &\qquad \leq C(\|\tilde{\phi}^k\|_{H^1}^2 + \|\tilde{\mathbf{u}}^k\|^2)C_0^4(\Delta t)^4 + C(\Delta t)^2 + \frac{1}{4}|\tilde{e}_R^{k+1}|^2. \end{aligned} \tag{4.65}$$

Since

$$\begin{aligned} &\Delta t \left| \frac{\tilde{R}^{k+1}}{E(\tilde{\phi}^{k+1}, \tilde{\mathbf{u}}^{k+1}) + \kappa_0} \Big(M\|\nabla\tilde{\mu}^{k+1}\|^2 + \nu\|\nabla\mathbf{u}^k\|^2\Big) - \Big(M\|\nabla\mu(t^{k+1})\|^2 + \nu\|\nabla\mathbf{u}(t^{k+1})\|^2\Big) \right| \\ &= \Delta t \left| \frac{\tilde{R}^{k+1}}{E(\tilde{\phi}^{k+1}, \tilde{\mathbf{u}}^{k+1}) + \kappa_0} - \frac{r(t^{k+1})}{E(\phi(t^{k+1}), \mathbf{u}(t^{k+1})) + \kappa_0} \right| \\ &\quad \Big(M\|\nabla\mu(t^{k+1})\|^2 + \nu\|\nabla\mathbf{u}(t^{k+1})\|^2\Big) \\ &\quad + M\Delta t \frac{\tilde{R}^{k+1}}{E(\tilde{\phi}^{k+1}, \tilde{\mathbf{u}}^{k+1}) + \kappa_0} (\|\nabla\tilde{\mu}^{k+1}\|^2 - \|\nabla\mu(t^{k+1})\|^2) \\ &\quad + \nu\Delta t \frac{\tilde{R}^{k+1}}{E(\tilde{\phi}^{k+1}, \tilde{\mathbf{u}}^{k+1}) + \kappa_0} (\|\nabla\mathbf{u}^k\|^2 - \|\nabla\mathbf{u}(t^{k+1})\|^2) \\ &\leq C\Delta t|\tilde{e}_R^{k+1}| + C\Delta t(\|\nabla\tilde{\mu}^{k+1}\| + 1)\|\nabla\tilde{e}_\mu^{k+1}\| + C\Delta t\|\tilde{e}_\mathbf{u}^k\| + C\Delta t. \end{aligned} \tag{4.66}$$

Thus using Cauchy-Schwarz inequality, the last term on the right hand side of (4.63) can be bounded by

$$\begin{aligned} H(\tilde{R}^{k+1},\tilde{\mu}^{k+1},\mathbf{u}^k) \le & C\Delta t|\tilde{e}_R^{k+1}|^2+\frac{1}{4K_2}\Delta t\|\nabla\tilde{\mu}^{k+1}\|^2|\tilde{e}_R^{k+1}|^2 \\ & +C\Delta t\|\nabla\tilde{e}_\mu^{k+1}\|^2+C\Delta t\|\tilde{e}_{\mathbf{u}}^k\|^2+C(\Delta t)^2. \end{aligned} \tag{4.67}$$

Combining (4.63) with (4.64)–(4.67), we have

$$\begin{aligned} |\tilde{e}_R^{k+1}|^2 \le & C(\|\tilde{\phi}^k\|_{H^1}^2+\|\tilde{\mathbf{u}}^k\|^2)C_0^4(\Delta t)^4+\frac{1}{4}|\tilde{e}_R^{k+1}|^2+C\Delta t|\tilde{e}_R^{k+1}|^2 \\ & +\frac{1}{4K_2}\Delta t\|\nabla\tilde{\mu}^{k+1}\|^2|\tilde{e}_R^{k+1}|^2+C\Delta t\|\nabla\tilde{e}_\mu^{k+1}\|^2+C\Delta t\|\tilde{e}_{\mathbf{u}}^k\|^2+C(\Delta t)^2. \end{aligned} \tag{4.68}$$

By substituting n for k and adding some positive terms on the right hand side of (4.68), we get

$$\begin{aligned} |\tilde{e}_R^{n+1}|^2 \le & C(\|\tilde{\phi}^n\|_{H^1}^2+\|\tilde{\mathbf{u}}^n\|^2)C_0^4(\Delta t)^4+\frac{1}{4}|\tilde{e}_R^{n+1}|^2+C_3\Delta t\sum_{k=0}^{n}|\tilde{e}_R^{k+1}|^2 \\ & +\frac{1}{4K_2}|\tilde{e}_R^{n+1}|^2\Delta t\sum_{k=0}^{n}\|\nabla\tilde{\mu}^{k+1}\|^2+C\Delta t\sum_{k=0}^{n}\|\nabla\tilde{e}_\mu^{k+1}\|^2+C\Delta t\sum_{k=0}^{n}\|\tilde{e}_{\mathbf{u}}^k\|^2+C(\Delta t)^2. \end{aligned} \tag{4.69}$$

Case II: Under the condition that $R^{k-1} < E(\phi^k,\mathbf{u}^k)+\kappa_0$, we have from (4.1), that $\sigma^k = 1-\frac{\tilde{R}^k(M\|\nabla\tilde{\mu}^k\|^2+\nu\|\nabla\mathbf{u}^{k-1}\|^2)}{(E(\tilde{\phi}^k,\tilde{\mathbf{u}}^k)+\kappa_0)(E(\phi^k,\mathbf{u}^k)+\kappa_0-\tilde{R}^k)}(\Delta t)^2$. Thus using (4.62), we have

$$\begin{aligned} \tilde{e}_R^{k+1}-\tilde{e}_R^k = & -\Delta t\frac{\tilde{R}^{k+1}}{E(\tilde{\phi}^{k+1},\tilde{\mathbf{u}}^{k+1})+\kappa_0}\left(M\|\nabla\tilde{\mu}^{k+1}\|^2+\nu\|\nabla\mathbf{u}^k\|^2\right) \\ & +\Delta t\frac{r(t^{k+1})}{E(\phi(t^{k+1}),\mathbf{u}(t^{k+1}))+\kappa_0}\left(M\|\nabla\mu(t^{k+1})\|^2+\nu\|\nabla\mathbf{u}(t^{k+1})\|^2\right) \\ & +\Delta t S_r^{k+1}+\frac{\tilde{R}^k\left(M\|\nabla\tilde{\mu}^k\|^2+\nu\|\nabla\mathbf{u}^{k-1}\|^2\right)}{E(\tilde{\phi}^k,\tilde{\mathbf{u}}^k)+\kappa_0}(\Delta t)^2. \end{aligned} \tag{4.70}$$

Multiplying (4.70) with $\tilde{e}_R^{k+1}$ and using the triangle inequality, we have

$$\begin{aligned} & \frac{1}{2}|\tilde{e}_R^{k+1}|^2-\frac{1}{2}|\tilde{e}_R^k|^2+\frac{1}{2}|\tilde{e}_R^{k+1}-\tilde{e}_R^k|^2 \\ & =\Delta t\left(\frac{\tilde{R}^k\left(M\|\nabla\tilde{\mu}^k\|^2+\nu\|\nabla\mathbf{u}^{k-1}\|^2\right)}{E(\tilde{\phi}^k,\tilde{\mathbf{u}}^k)+\kappa_0}\Delta t,\tilde{e}_R^{k+1}\right)+\Delta t(S_r^{k+1},\tilde{e}_R^{k+1}) \\ & \quad +H(\tilde{R}^{k+1},\tilde{\mu}^{k+1},\mathbf{u}^k), \end{aligned} \tag{4.71}$$

Taking the sum of (4.71) from $q+1$ to n with $\sigma^q=0$ and using (4.63) result in

$$\begin{aligned} \frac{1}{2}|\tilde{e}_R^{n+1}|^2 \le & \Delta t\sum_{k=q+1}^{n}\left(\frac{\tilde{R}^k\left(M\|\nabla\tilde{\mu}^k\|^2+\nu\|\nabla\mathbf{u}^{k-1}\|^2\right)}{E(\tilde{\phi}^k,\tilde{\mathbf{u}}^k)+\kappa_0}\Delta t,\tilde{e}_R^{k+1}\right) \\ & +\frac{1}{2}\left(E(\phi^q,\mathbf{u}^q)-E(\phi(t^q),\mathbf{u}(t^q)),\tilde{e}_R^{q+1}\right)+\frac{1}{2}\Delta t(S_r^{q+1},\tilde{e}_R^{q+1}) \\ & +\Delta t\sum_{k=q+1}^{n}(S_r^{k+1},\tilde{e}_R^{k+1})+\frac{1}{2}H(\tilde{R}^{q+1},\tilde{\mu}^{q+1},\mathbf{u}^q)+\sum_{k=q+1}^{n}H(\tilde{R}^{k+1},\tilde{\mu}^{k+1},\mathbf{u}^k). \end{aligned} \tag{4.72}$$

Using (4.14) and lemma 4.2, the first term on the right hand side of (4.72) can be estimated as

$$\Delta t\sum_{k=q+1}^{n}\left(\frac{\tilde{R}^k\left(M\|\nabla\tilde{\mu}^k\|^2+\nu\|\nabla\mathbf{u}^{k-1}\|^2\right)}{E(\tilde{\phi}^k,\tilde{\mathbf{u}}^k)+\kappa_0}\Delta t,\tilde{e}_R^{k+1}\right)\leq\frac{1}{4}|\tilde{e}_R^{n+1}|^2+C(K_1+K_2)(\Delta t)^2, \tag{4.73}$$

where we suppose that $|\tilde{e}_R^{n+1}|^2$ achieves its maximum value at the time step $n+1$. Otherwise we should repeat this estimate.

The other terms on the right hand side of (4.72) can be estimated by using exactly the same procedure as above in Case I, we can obtain that

$$\begin{aligned}|\tilde{e}_R^{n+1}|^2\leq&C_4\Delta t\sum_{k=q+1}^{n}|\tilde{e}_R^{k+1}|^2+C\Delta t\sum_{k=0}^{n}\|\nabla\tilde{e}_\mu^{k+1}\|^2\\&+C\Delta t\sum_{k=0}^{n}\|\tilde{e}_\mathbf{u}^k\|^2+C(\|\tilde{\phi}^q\|_{H^1}^2+\|\tilde{\mathbf{u}}^q\|^2)C_0^4(\Delta t)^4+C(\Delta t)^2.\end{aligned} \tag{4.74}$$

Applying the discrete Gronwall lemma 2.3 under the condition that $\Delta t\leq\min\{\frac{1}{4C_3},\frac{1}{4C_4}\}$ and using (4.58), we can arrive at

$$|\tilde{e}_R^{n+1}|^2\leq C_5\left(1+C_0^4(\Delta t)^2\right)(\Delta t)^2,\ \forall n\leq T/\Delta t. \tag{4.75}$$

Next we finish the induction process as follows. Recalling (3.7), we have

$$\begin{aligned}|1-\xi^{n+1}|=&\left|\frac{R(t^{n+1})}{E(\phi(t^{n+1}),\mathbf{u}(t^{n+1}))+\kappa_0}-\frac{\tilde{R}^{n+1}}{E(\tilde{\phi}^{n+1},\tilde{\mathbf{u}}^{n+1})+\kappa_0}\right|\\\leq&C(|\tilde{e}_R^{n+1}|+\|\tilde{e}_\mathbf{u}^{n+1}\|+\|\tilde{e}_\phi^{n+1}\|+\|\nabla\tilde{e}_\phi^{n+1}\|)\\\leq&C_6\Delta t\sqrt{1+C_0^4(\Delta t)^2},\quad\forall n\leq T/\Delta t.\end{aligned} \tag{4.76}$$

where C_6 is independent of C_0 and Δt.

Let $C_0=\max\{2C_6,2\sqrt{C_4},2\sqrt{C_3},(2C_2)^{\frac{3}{2}},\sqrt{\frac{2C_1}{\lambda}},4\}$ and $\Delta t\leq\frac{1}{1+C_0^2}$, we can obtain

$$C_6\sqrt{1+C_0^4(\Delta t)^2}\leq C_6(1+C_0^2\Delta t)\leq C_0. \tag{4.77}$$

Then combining (4.76) with (4.77) results in

$$|1-\xi^{n+1}|\leq C_0\Delta t,\ \forall n\leq T/\Delta t. \tag{4.78}$$

Recalling (4.58), we have

$$\|\tilde{e}_\mathbf{u}^{n+1}\|_{H^2}+\|\tilde{e}_\mu^{n+1}\|_{H^1}\leq C_2\left(1+C_0^4(\Delta t)^2\right)(\Delta t)^{1/2}\leq(\Delta t)^{1/6}, \tag{4.79}$$

which completes the induction process (4.9) and (4.10).

Then combining (4.58) with (4.49), (4.52) and (4.53), we obtain

$$\begin{aligned}&\|e_\phi^{n+1}\|^2+\|\nabla e_\phi^{n+1}\|^2+\Delta t\sum_{k=0}^{n}\|e_\mu^{k+1}\|^2+\Delta t\sum_{k=0}^{n}\|\nabla e_\mu^{k+1}\|^2\\&+\|\nabla e_\mathbf{u}^{n+1}\|^2+\Delta t\sum_{k=0}^{n}\|\Delta e_\mathbf{u}^{k+1}\|^2\leq C(\Delta t)^2,\ \forall n\leq T/\Delta t.\end{aligned} \tag{4.80}$$

Now it remains to estimate the pressure error. Recalling (4.46), we can estimate (4.45) into the following:

$$\begin{aligned}
\Delta t\sum_{k=0}^{n}\|\nabla e_p^{k+1}\|^2 &\le C\Delta t\sum_{k=0}^{n}\|\Delta\tilde{e}_{\mathbf{u}}^{k+1}\|^2+C\Delta t\sum_{k=0}^{n}\|\nabla\tilde{e}_{\mathbf{u}}^{k+1}\|^2\\
&+C\Delta t\sum_{k=0}^{n}(\|\nabla e_{\mathbf{u}}^{k+1}\|^2\|\nabla\mathbf{u}(t^{k+1})\|\|\nabla\mathbf{u}(t^{k+1})\|_{H^1}+\|\nabla\mathbf{u}^{k+1}\|^2\|\nabla e_{\mathbf{u}}^{k+1}\|\|\nabla e_{\mathbf{u}}^{k+1}\|_{H^1})\\
&+C\Delta t\sum_{k=0}^{n}(\|\nabla e_{\mu}^{n+1}\|^2\|\nabla\phi^{n+1}\|\|\nabla\phi^{n+1}\|_{H^1}+\|\nabla\mu(t^{n+1})\|^2\|\nabla e_{\phi}^{n+1}\|\|\nabla e_{\phi}^{n+1}\|_{H^1})\\
&\le C(\Delta t)^2,\ \forall n\le T/\Delta t,
\end{aligned} \tag{4.81}$$

which completes the proof. □

5 Numerical Experiments

In this section, we provide some numerical experiments to verify our theoretical results of the constructed high-order GSAV schemes with the new relaxation (3.4)–(3.11) for the Cahn–Hilliard–Navier–Stokes model.

5.1 Convergence Tests

We first verify the accuracy of the proposed numerical schemes. We choose the coefficients

$$\lambda=1,\quad M=1\times10^{-3},\quad \epsilon=1,\quad \gamma=0,\quad \nu=0.05, \tag{5.1}$$

and solve (1.1) with right hand sides chosen so that the exact solution is

$$\begin{aligned}
&\phi(x,y,t)=\cos(t)\cos(\pi x)\cos(\pi y),\\
&\mathbf{u}(x,y,t)=\pi\sin(t)(\sin^2(\pi x)\sin(2\pi y),-\sin(2\pi x)\sin^2(\pi y))^T,\\
&p(x,y,t)=\sin(t)\cos(\pi x)\sin(\pi y).
\end{aligned} \tag{5.2}$$

We set $\Omega=(-1,1)^2$ and use 50×50 modes to discretize the space variables, so the spatial discretization error is negligible compared to the time discretization. In Fig. 1(a–e), we list the errors between the numerical solution and the exact solution at $T=0.2$. We observe that all schemes achieve the expected accuracy in time, which is consistent with the error analysis in Theorem 4.1.

5.2 Shape Relaxation

In this simulation, we employ an evolution of a star-shaped interface, with the initial value provided by

$$\begin{aligned}
&\phi(x,y,0)=\tanh\frac{0.25+0.1\cos(s\theta+\pi/2)-r}{\sqrt{2}\epsilon},\ \mathbf{u}(x,y,0)=\mathbf{0},\\
&\theta=\arctan\frac{y-0.5}{x-0.5},\quad r=\sqrt{(x-0.5)^2+(y-0.5)^2},
\end{aligned}$$

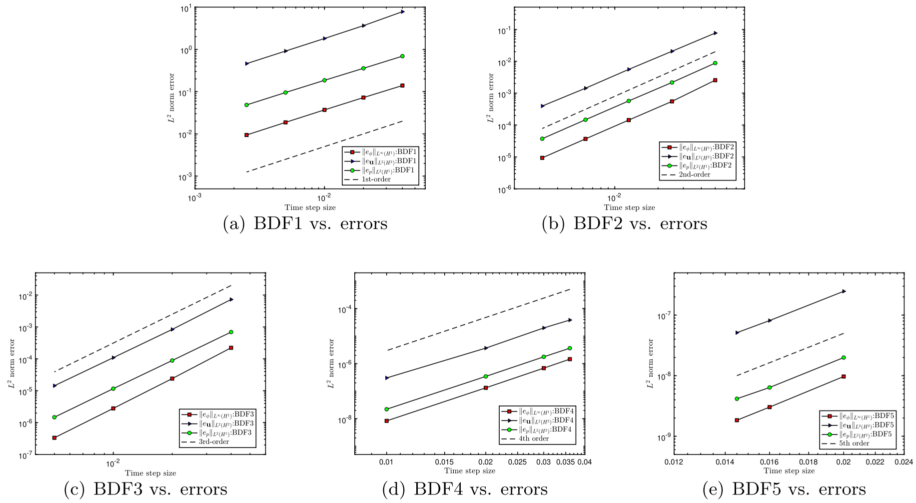


Fig. 1 Numerical convergence rates of the first- to fifth-order schemes

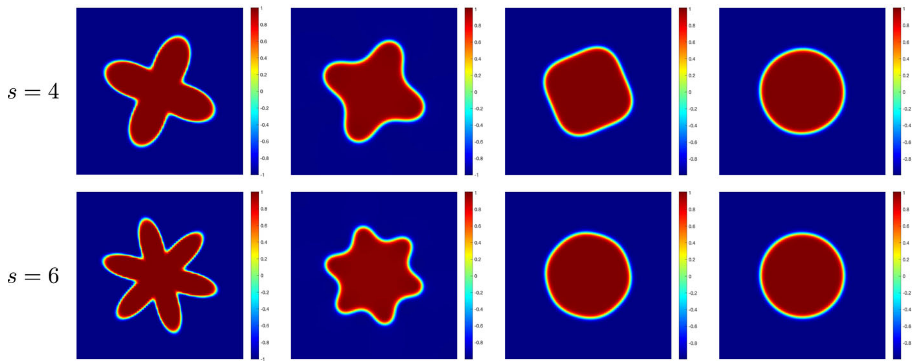


Fig. 2 Snapshots of the phase function ϕ at $t = 0,\ 0.2,\ 0.6,\ 1.5$

where the parameter s denotes the count of vertices in the initial data. We set $\Omega = (0, 1)^2$ and use 128×128 modes to discretize the space variables. The parameters are

$$\begin{aligned} \Delta t &= 1 \times 10^{-3}, \quad \lambda &= 1 \times 10^{-2}, \quad M = 1 \times 10^{-3}, \\ \epsilon &= 1 \times 10^{-2}, \quad \gamma &= 2 \times 10^{4}, \quad \nu = 1. \end{aligned} \tag{5.3}$$

In Fig. 2, we depict the dynamic process of shape relaxation towards a disk, considering various initial values with third-order scheme. It can be clearly observed that the energy dissipation law holds for both $s = 4$ and $s = 6$ in Fig. 3 (Left). Furthermore, we can also find that the more vertices, the faster the evolution by comparing the energy evolutions of two cases.

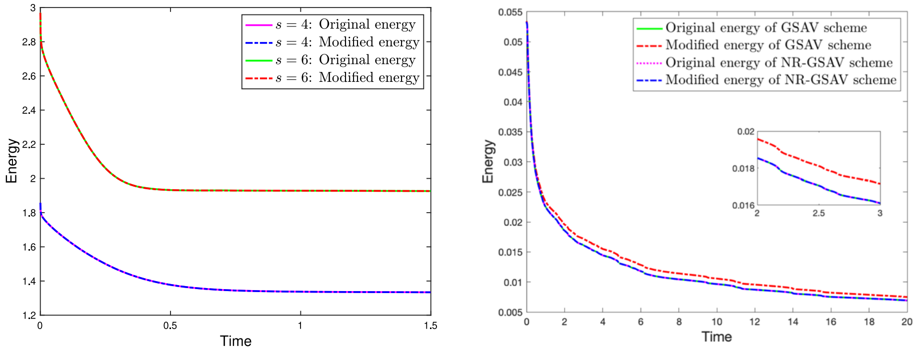


Fig. 3 Evolutions of both the original and modified energy curves under shape relaxation (Left), as presented in subsection 5.2. The energy curves that depict the nucleation process (Right) using the GSAV and NR-GSAV schemes can be found in subsection 5.3

5.3 Flow-Coupled Phase Separation

In this subsection, the process of flow-coupled nucleation is considered. The initial conditions are as follows

$$\phi(x, y, 0) = y - 1 + 0.01\text{rand}(x, y), \quad \mathbf{u}(x, y, 0) = \mathbf{0},$$

where $\text{rand}(x, y)$ represents the random distribution between -1 and 1.

We set $\Omega = (0, 2)^2$ and use 200×200 modes to discretize the space variables and the other parameters are set as follows:

$$\begin{aligned} \Delta t &= 1 \times 10^{-3}, \quad \lambda = 1 \times 10^{-5}, \quad M = 5 \times 10^{-1}, \\ \epsilon &= 1 \times 10^{-2}, \quad \gamma = 2 \times 10^{4}, \qquad \nu = 1. \end{aligned} \tag{5.4}$$

The magnitude of ϕ has a larger value close to the upper and lower boundaries and a smaller value close to the domain center with the specified initially condition. We observe a monotonic decay of energy with third-order scheme in Fig. 3 (Right), with the original and modified energy of the GSAV scheme with the new relaxation (NR-GSAV) showing enhanced consistency compared to the GSAV scheme over the same time step. In Figrue 4, it can be noticed that the phase separation occurs close to the domain center. The generated droplets eventually disappear over time since the interfacial length as a whole shrank as a result of energy decays.

5.4 Buoyancy-Driven Flow

In this example, we reformulate the momentum equation as follows:

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} - \nu \Delta \mathbf{u} + \nabla p = \mu \nabla \phi + \chi \rho(\phi) \mathbf{g}, \tag{5.5}$$

where $\chi \rho(\phi) \mathbf{g}$ is a buoyancy term with $\chi \rho(\phi) = \chi(\phi - \bar{\phi})$, and $\bar{\phi}$ is spatially averaged order parameter. The computational domain is $\Omega = (0, 1)^2$. We use 128×128 modes to discretize the space variables and the initial velocity is initialized as $\mathbf{u}(x, y, 0) = \mathbf{0}$.

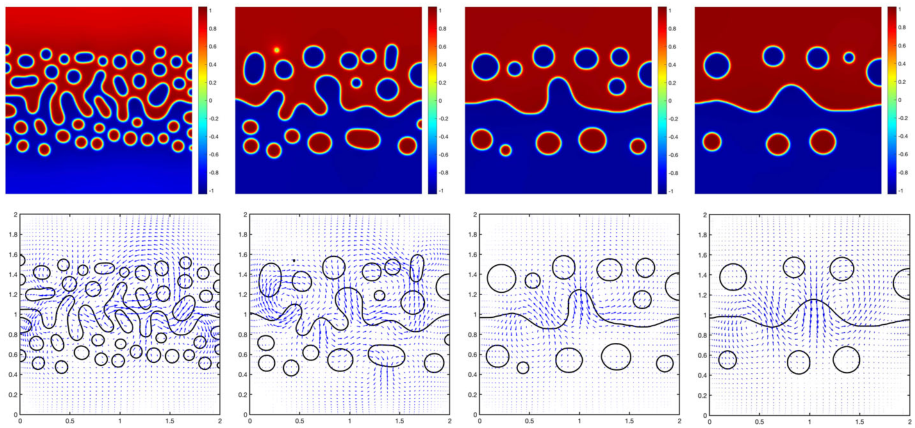


Fig. 4 The top and bottom rows display the snapshots of the phase function ϕ and velocity field for the flow-coupled nucleation process at time $t = 1,\ 5,\ 10,\ 20$

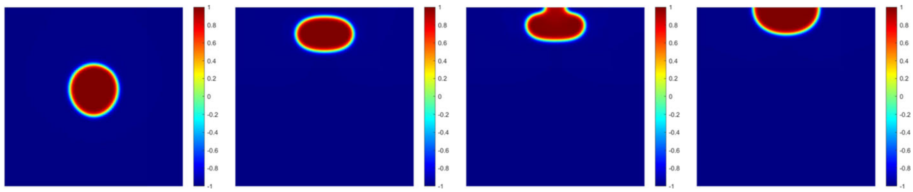


Fig. 5 Snapshots of the phase function ϕ at $t = 1,\ 2.55,\ 3.25,\ 4$

5.4.1 Bubble Rising

The numerical and physical parameters are provided as follows:

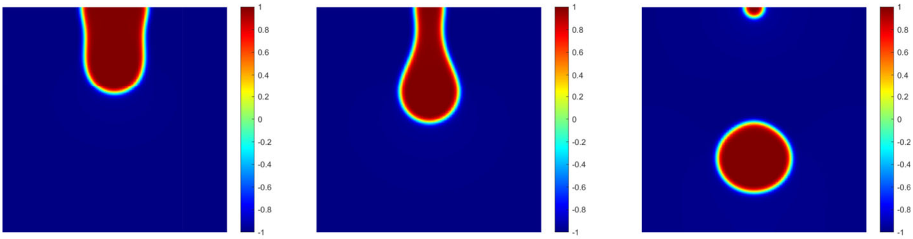
$$\begin{aligned} \Delta t =& 5 \times 10^{-4}, \quad \lambda = 1 \times 10^{-3}, \quad M = 1 \times 10^{-2}, \quad \epsilon = 1 \times 10^{-2}, \\ \gamma =& 2 \times 10^{4}, \quad \nu = 1, \quad g = (0, -1)^T, \quad \chi = 5 \times 10. \end{aligned} \tag{5.6}$$

We set the initial condition for the phase function as a circular bubble centered at (0.5, 0.25) with a radius of $r = 0.15$. In Fig. 5, snapshots of the phase evolution with second-order scheme at different times ($t = 1, 1.75, 2.55, 3.25, 3.35, 4$) are displayed. Initially, the bubble appears as a circular shape near the bottom of the domain. The bubble, which is lighter compared to the surrounding fluid, rises gradually, transitioning into an elliptical shape, and eventually deforms as it approaches the upper boundary, as expected.

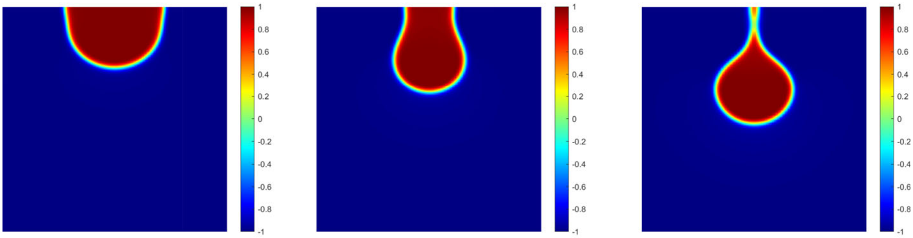
5.4.2 Dripping Droplet

We first conduct simulations to observe the evolving behavior of a dripping droplet under different Reynolds numbers: $\nu = 1/10$, $1/100$, and $1/200$. Set the parameters as

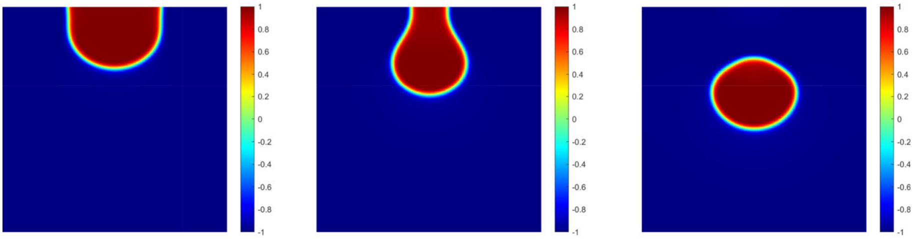
$$\begin{aligned} \Delta t =& 1 \times 10^{-3}, \quad \lambda = 1 \times 10^{-3}, \quad M = 1 \times 10^{-2}, \quad \epsilon = 1 \times 10^{-2}, \\ \gamma =& 2 \times 10^{4}, \quad g = (0, 1)^T, \quad \chi = 10. \end{aligned} \tag{5.7}$$



(a) $\nu = 0.1,\ T{=}0.8$ (b) $\nu = 0.1,\ T{=}1.2$ (c) $\nu = 0.1,\ T{=}2$



(d) $\nu = 0.01,\ T{=}0.2$ (e) $\nu = 0.01,\ T{=}0.4$ (f) $\nu = 0.01,\ T{=}0.6$



(g) $\nu = 0.005,\ T{=}0.2$ (h) $\nu = 0.005,\ T{=}0.4$ (i) $\nu = 0.005,\ T{=}0.6$

Fig. 6 Snapshots of the phase function ϕ at different T

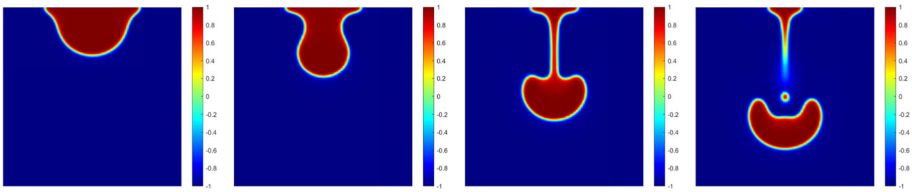


Fig. 7 Snapshots of the phase function ϕ at $t = 0.2,\ 0.4,\ 0.7,\ 0.9$

The initial condition is as follows

$$\phi(x, y, 0) = \tanh\left(\frac{0.32 - \sqrt{(x-0.5)^2 + (y-1.1)^2}}{\sqrt{2}\epsilon}\right).$$

Initially, the droplet with heavier density is attached to the upper solid wall. Due to the influence of gravity, the droplet gradually descends over time. In Fig. 6, the top row presents a snapshot of the droplet at $\nu = 1/10$, while the second and third rows depict the evolution results for $\nu = 1/100$ and $\nu = 1/200$ respectively. It is evident that as the Reynolds number $1/\nu$ increases, the pinch off occurs more rapidly and the droplet descends at a faster rate.

Next, we choose the parameters as

$$\begin{aligned} \Delta t = & 1 \times 10^{-3}, \quad \lambda = 1 \times 10^{-5}, \quad M = 1 \times 10^{-1}, \quad \epsilon = 7.5 \times 10^{-3}, \\ \gamma = & \frac{2}{\epsilon^2}, \quad \nu = 5 \times 10^{-2}, \quad g = (0,1)^T, \quad \chi = 10. \end{aligned} \tag{5.8}$$

The snapshots of the droplet are shown in Fig. 7 with the initial state unchanged. The formation of spike structures becomes evident, particularly when the liquid filament is extremely elongated.

6 Concluding Remarks

In this paper we constructed novel fully decoupled and higher-order IMEX schemes for the Cahn–Hilliard–Navier–Stokes system based on the GSAV method with the new relaxation for Cahn–Hilliard equation and the consistent splitting method for Navier–Stokes equations. The resulting higher-order schemes are fully decoupled, linear, unconditional energy stable and only require solving several Poisson type equations with constant coefficients at each time step. We also carried out a rigorous global-in-time error analysis for the first-order scheme in two and three-dimensional cases.

We only carried out the error analysis for the first-order scheme, due in principle to the lack of strong stability for the second- and higher-order consistent splitting schemes for the Navier–Stokes equations. Recently a consistent splitting scheme for the Navier–Stokes equations based on a generalized BDF was introduced in [16], and its second-order version was shown to possess a strong stability. While its error analysis for the Navier–Stokes equations is much more complicated, it offers potential for an error analysis of the scheme (3.4)–(3.11) at second- and higher-order discretizations. We plan to investigate this potential in a future work.

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Data Availability The data used to support the findings of this study are available from the corresponding author upon request.

Declarations

Conflict of interest The authors declare that they have no Conflict of interest.

References

1. Abels, H.: On a diffuse interface model for two-phase flows of viscous, incompressible fluids with matched densities. Arch. Ration. Mech. Anal. **194**, 463–506 (2009)
2. An, R., Gao, H., Sun, W.: Optimal error analysis of Euler and Crank-Nicolson projection finite difference schemes for Landau-Lifshitz equation. SIAM J. Numer. Anal. **59**, 1639–1662 (2021)
3. Brenner, S.C., Scott, L.R., Scott, L.R.: The mathematical theory of finite element methods, vol. 3. Springer (2008)

4. Du, Q., Feng, X.: The phase field method for geometric moving interfaces and their numerical approximations, In: Handbook of Numerical Analysis, vol. 21, Elsevier, pp. 425–508 (2020)
5. Eyre, D.J.: Unconditionally gradient stable time marching the Cahn-Hilliard equation, In: MRS Proceedings, vol. 529, Cambridge Univ Press, p. 39 (1998)
6. Feng, X., He, Y., Liu, C.: Analysis of finite element approximations of a phase field model for two-phase fluids. Math. Comput. **76**, 539–571 (2007)
7. Grün, G.: On convergent schemes for diffuse interface models for two-phase flow of incompressible fluids with general mass densities. SIAM J. Numer. Anal. **51**, 3036–3061 (2013)
8. Guermond, J., Shen, J.: A new class of truly consistent splitting schemes for incompressible flows. J. Comput. Phys. **192**, 262–276 (2003)
9. Guermond, J.L., Minev, P., Shen, J.: An overview of projection methods for incompressible flows. Comput. Methods Appl. Mech. Eng. **195**, 6011–6045 (2006)
10. Guermond, J.-L., Shen, J.: Velocity-correction projection methods for incompressible flows. SIAM J. Numer. Anal. **41**, 112–134 (2003)
11. Gurtin, M.E., Polignone, D., Vinals, J.: Two-phase binary fluids and immiscible fluids described by an order parameter. Math. Models Methods Appl. Sci. **6**, 815–831 (1996)
12. Han, D., Brylev, A., Yang, X., Tan, Z.: Numerical analysis of second order, fully discrete energy stable schemes for phase field models of two-phase incompressible flows. J. Sci. Comput. **70**, 965–989 (2017)
13. Han, D., Wang, X.: A second order in time, uniquely solvable, unconditionally stable numerical scheme for Cahn–Hilliard-Navier–Stokes equation. J. Comput. Phys. **290**, 139–156 (2015)
14. Huang, F., Shen, J.: Stability and error analysis of a class of high-order IMEX schemes for Navier–Stokes equations with periodic boundary conditions. SIAM J. Numer. Anal. **59**, 2926–2954 (2021)
15. Huang, F., Shen, J.: A new class of implicit-explicit BDFk SAV schemes for general dissipative systems and their error analysis. Comput. Methods Appl. Mech. Eng. **392**, 114718 (2022)
16. Huang, F., Shen, J.: Stability and error analysis of a second-order consistent splitting scheme for the Navier–Stokes equations. SIAM J. Numer. Anal. **61**, 2408–2433 (2023)
17. Li, X., Shen, J.: On a SAV-MAC scheme for the Cahn–Hilliard-Navier–Stokes phase field model and its error analysis for the corresponding Cahn-Hilliard-Stokes case. Math. Models Methods Appl. Sci. **30**, 2263–2297 (2020)
18. Li, X., Shen, J.: On fully decoupled msav schemes for the Cahn–Hilliard-Navier–Stokes model of two-phase incompressible flows. Math. Models Methods Appl. Sci. **32**, 457–495 (2022)
19. Li, X., Shen, J., Liu, Z.: New SAV-pressure correction methods for the Navier–Stokes equations: stability and error analysis. Math. Comput. **91**, 141–167 (2022)
20. Liu, C., Shen, J.: A phase field model for the mixture of two incompressible fluids and its approximation by a Fourier-spectral method. Phys. D **179**, 211–228 (2003)
21. Liu, J.-G., Liu, J., Pego, R.L.: Stability and convergence of efficient Navier–Stokes solvers via a commutator estimate, Communications on Pure and Applied Mathematics: A Journal Issued by the Courant Institute of. Math. Sci. **60**, 1443–1487 (2007)
22. Serson, D., Meneghini, J., Sherwin, S.J.: Velocity-correction schemes for the incompressible Navier–Stokes equations in general coordinate systems. J. Comput. Phys. **316**, 243–254 (2016)
23. Shen, J.: Long time stability and convergence for fully discrete nonlinear Galerkin methods. Appl. Anal. **38**, 201–229 (1990)
24. Shen, J.: On error estimates of projection methods for Navier–Stokes equations: first-order schemes. SIAM J. Numer. Anal. **29**, 57–77 (1992)
25. Shen, J., Xu, J.: Convergence and error analysis for the scalar auxiliary variable (SAV) schemes to gradient flows. SIAM J. Numer. Anal. **56**, 2895–2912 (2018)
26. Shen, J., Xu, J., Yang, J.: A new class of efficient and robust energy stable schemes for gradient flows. SIAM Rev. **61**, 474–506 (2019)
27. Shen, J., Yang, X.: Numerical approximations of Allen-Cahn and Cahn-Hilliard equations. Discrete Contin. Dyn. Syst **28**, 1669–1691 (2010)
28. Shen, J., Yang, X.: A phase-field model and its numerical approximation for two-phase incompressible flows with different densities and viscosities. SIAM J. Sci. Comput. **32**, 1159–1179 (2010)
29. Shen, J., Yang, X.: Decoupled, energy stable schemes for phase-field models of two-phase incompressible flows. SIAM J. Numer. Anal. **53**, 279–296 (2015)
30. Shen, J., Yang, X.: The IEQ and SAV approaches and their extensions for a class of highly nonlinear gradient flow systems, In: 75 years of mathematics of computation, vol. 754 of Contemp. Math., Amer. Math. Soc., Providence, RI, pp. 217–245 (2020)
31. Yang, X.: Linear, first and second-order, unconditionally energy stable numerical schemes for the phase field model of homopolymer blends. J. Comput. Phys. **327**, 294–316 (2016)

32. Weinan, E., Liu, J.-G.: Gauge method for viscous incompressible flows. Commun. Math. Sci. **1**, 317–332 (2003)