MA 584 HW 4 DUE OCTOBER 16TH

If K is a field with nonarchimedean valuation $|\ |_v$ then define

$$\mathcal{O}_{(v)} = \{x \in K |\ |x|_v \leq 1\}$$

and

$$\mathfrak{P}_{(v)} = \{x \in K |\ |x|_v < 1\}.$$

(1) (a) Show that $\mathcal{O}_{(v)}$ is integrally closed.
 (b) We have shown that $(\mathcal{O}_{(v)}, \mathfrak{P}_{(v)})$ is a local ring. Furthermore, in many cases (e.g., K is a number field), $(\mathcal{O}_{(v)}, \mathfrak{P}_{(v)})$ is just a DVR. Is this always true in general?
 (c) Let K_v denote the completion under the valuation $|\ |_v$. Define

$$\mathcal{O}_v = \{x \in K_v |\ |x|_v \leq 1\}$$

and

$$\mathfrak{P}_v = \{x \in K |\ |x|_v < 1\}.$$

show that $\mathcal{O}_v$ is the completion of $\mathcal{O}_{(v)}$, $\mathfrak{P}_v$ is the completion of $\mathfrak{P}_{(v)}$ and $\mathcal{O}_v/\mathfrak{P}_v \simeq \mathcal{O}_{(v)}/\mathfrak{P}_{(v)}$.
 (d) Further assume that K is a number field. Write $\mathcal{O} = \mathcal{O}_K$ the ring of algebraic integers. Show that $\mathcal{O}_v$ is the completion of $\mathcal{O}$.

(2) Let us discuss valuation of function field. Let K be field over $\mathbb{F}_p := \mathbb{Z}/p\mathbb{Z}$.
 (a) Prove if K admits a valuation then it is must be nonarchimdean.
 (b) Determine all valuation of k where k is a finite extension of $\mathbb{F}_p$.
 (c) Determine all valuation $|\ |_v$ of $K = \mathrm{Frac}(k[u])$ so that $|u|_v \leq 1$.
 (d) Let $k = \overline{\mathbb{F}}_p$ and $|\ |_v$ is a valuation of $K = \mathrm{Frac}(k[u])$ so that $|u|_v \leq 1$. Show that $\mathcal{O}_v \simeq k[\![t]\!]$.

(3) It is important to realize p-adic number as projective limit: Recall there exists a natural projection $\mathbb{Z}/p^{n+1}\mathbb{Z} \twoheadrightarrow \mathbb{Z}/p^n\mathbb{Z}$ induced by modulo p^n. Show that

$$\mathbb{Z}_p \simeq \varprojlim \mathbb{Z}/p^n\mathbb{Z}$$

as topological rings. Recall that the topology of right side uses finite index subgroups as a basis of open sets. Indeed, this is valid for $\mathcal{O}_K$ and $\mathfrak{P} \in \mathrm{Spec}(\mathcal{O}_K)$ for K being a number field. We have

$$\mathcal{O}_{K,v} \simeq \varprojlim \mathcal{O}_K/\mathfrak{P}^n$$

where $|\ |_v$ the valuation induced by $\mathfrak{P}$.