MA 584 HW 6 DUE NOV. 13TH

In the following, A will be always a Dedekind domain and $K = \mathrm{Frac}(A)$. Let E/K be a finite separable extension. For any $\mathfrak{p}$ a prime a A, always assume $k = A/\mathfrak{p}$ is perfect, so that any finite extension of k is separable.

(1) Let $\mathfrak{a}$ be a fractional ideal of B. In general, we can not expect that $\mathfrak{a}$ is a finite free A-module. Given a K-basis $w_1, \dots, w_n$ of E. Show that there exists a nonzero element $b \in A$ so that
$$b(Aw_1 + \cdots + Aw_n) \subset \mathfrak{a} \subset \frac{1}{b}(Aw_1 + \cdots + Aw_n).$$

(2) (This is a wrong statement and example found) Let $S = A - \mathfrak{p}$, $A_{(\mathfrak{p})} := S^{-1}A$ and $B_{(\mathfrak{p})} = S^{-1}B$. Show that there exists an $\alpha \in B$ so that $B_{(\mathfrak{p})} = A_{(\mathfrak{p})}[\alpha]$. (Hint: For each prime $\mathfrak{P}$ above $\mathfrak{p}$, there exists an $\alpha_{\mathfrak{P}} \in B_{(\mathfrak{P})}$ such that $B_{(\mathfrak{P})} = A_{(\mathfrak{p})}[\alpha_{\mathfrak{P}}]$ by Prop. 3. Choose $\alpha \in B$ so that $|\alpha - \alpha_{\mathfrak{P}}|_{\mathfrak{P}}$ is very small for all $\mathfrak{P}$. Show that for any $x \in B$ there exists $y \in A[\alpha]$ so that $y - x \in \mathfrak{p}B$ and then use NAK to prove $A_{(\mathfrak{p})}[\alpha] = B_{(\mathfrak{p})}$.)

(3) (a) Suppose $K = \mathbb{Q}$ and $A = \mathbb{Z}$. Let $\alpha \in B = \mathcal{O}_E$. If $E = \mathbb{Q}(\alpha)$ and $D_{E/\mathbb{Q}}(\alpha)$ is a *square free* integer then show that $B = A[\alpha]$.
(b) Let $E = \mathbb{Q}(\alpha)$ where α is a root of $X^3 + X + 1$. Show that $B = A[\alpha]$.

(4) Let $E = \mathbb{Q}(\sqrt{D})$ be a quadratic field, where D is a square free integer.
(a) Compute the discriminant $D_{E/\mathbb{Q}}$ (Hint: you can use results of previous HWs).
(b) When $p = 2$ is ramified over E?