MATH 341, Exam #1 Key

Problem 1. Let $a_0 \geqslant 1$ and $a_{n+1} = 1 + \ln a_n$. Show that the sequence (a_n) converges. Hint: $x < e^x/e$ for $x > 1$.

$a_{n+1} = 1 + \ln(a_n) < 1 + \ln(e^{a_n}/e) = 1 + a_n - 1 = a_n$, so decreasing (2 pts). $a_0 \geqslant 1$ and $a_n \geqslant 1$ implies $\ln(a_n) \geqslant 0$, so $a_{n+1} \geqslant 1$. The sequence is thus bounded below (2 pts.) thus converges by completeness (1 pt.).

Problem 2. Show that the sequence $I_n = \int_0^1 3^n x^n (1-x)^n dx$ converges and find the limit.

$3x(1-x)$ is a downward parabola, positive on $[0,1]$ with roots at 0 and 1, so max is $3/4$ at $x = 1/2$ (2 pts.). Thus $3^n x^n (1-x)^n$ has maximum value $(3/4)^n$ (1 pt.). Thus $0 \leqslant I_n \leqslant (3/4)^n (1-0)$ (1 pts.) and $I_n \to 0$ by the Squeeze theorem (1 pt.)

Problem 3. Show that for any sequence (a_n), the sequence

$$b_n = \int_0^{|a_n|} \frac{1}{1+x^{2019}} dx$$

has a convergent subsequence.

If $x \geqslant 1$ then $\frac{1}{1+x^{2019}} \leqslant \frac{1}{1+x^2}$. Thus $0 \leqslant b_n \leqslant \int_0^1 1 dx + \int_0^{|a_n|} \frac{1}{1+x^2} dx \leqslant 1 + \pi/2$ (3 pts.). So (b_n) is bounded (1 pt.) and has a convergent subsequence by the Bolzano–Weierstrass Theorem (1 pt.).

Problem 4. Let (a_n) be a bounded decreasing sequence. Show that

$$\lim_{n\to\infty} a_n = \inf(\{a_1, a_2, \dots, \}).$$

If m is a lower bound, then $L = \lim_{n\to\infty} a_n \geqslant m$ by the limit location theorem (2 pts.). We have that a_n is an upper bound for a tail of (a_n), so $a_n \geqslant L$ by the same and L is a lower bound for $\{a_1, a_2, \dots, \}$ (2 pts.). Thus L is the greatest lower bound (1 pt.).

Problem 5. Define what it means for L to be a cluster point of a sequence (a_n). If A is a cluster point of (a_n) and B is a cluster point of (b_n), is $A+B$ a cluster point of (a_n+b_n)? Prove or give a counterexample.

A cluster point L is a number such for every $\epsilon > 0$, $|a_n - L| < \epsilon$ for infinitely many n (2 pts.). A counterexample is $(1,0,1,0,\dots)$ and $(0,1,0,1,\dots)$ both have 0 as a cluster point, but as the sum of the sequences is constant with value 1, the only cluster point is $1 \neq 0+0$. (3 pts.)

Problem 6. State the Nested Intervals Theorem. Use the Nested Interval Theorem directly (without using any other theorem) to show that every Cauchy sequence converges.

The Nested Intervals Theorem states that if $[a_1, b_1] \supset [a_2, b_2] \supset \cdots \supset [a_n, b_n] \supset \cdots$ is a nested collection of closed intervals such that $|a_n - b_n| \to 0$, then there is some number L such that $\{L\} = \bigcap_{n=1}^{\infty} [a_n, b_n]$ (2 pts.). Let (a_n) be a Cauchy sequence,. Let $A_n = \inf\{a_n, a_{n+1}, \dots\}$ and $B_n = \sup\{a_n, a_{n+1}, \dots\}$. We have that $A_n \leqslant A_{n+1} \leqslant$

$B_{n+1} \leqslant B_n$ for all n (2 pts.). Thus $[A_n, B_n]$ is a nested family of intervals and $|A_n - B_n| = \sup\{|a_k - a_\ell| : k, l \geqslant n\} \to 0$ by the Cauchy property. So the NIT applies and (a_n) is seen to converge to L (1 pt.)