MATH 341, EXAM #2 KEY

Problem 1. State the ϵ-δ definition of the limit $\lim_{x\to a} f(x) = L$. Compute

$$\lim_{x\to 1} \frac{x-1}{\sqrt{x}-1}$$

and verify your answer is correct by using the ϵ-δ definition of the limit.

Definition: $\forall \epsilon > 0$, $\exists \delta > 0$, so that $0 < |x - a| < \delta$ implies $|f(x) - L| < \epsilon$. (2 points)

$$\lim_{x\to 1} \frac{x-1}{\sqrt{x}-1} = \lim_{x\to 1} \frac{(\sqrt{x}-1)(\sqrt{x}+1)}{\sqrt{x}-1} = \lim_{x\to 1} \sqrt{x}+1 = 2 \text{ (1pt.)}$$

$$|\sqrt{x}+1-2| = |\sqrt{x}-1| = \frac{|x-1|}{\sqrt{x}+1} < |x-1|$$

if $x > 0$. Thus $\delta = \epsilon$ suffices (2 pts.)

Problem 2. Let f and g be functions defined on some neighborhood of 0. If $f(x) \to 0$ as $x \to 0$ and g is locally bounded at 0, show that the product fg is continuous at zero. Give an example of a function defined for all real numbers which is continuous at exactly two points.

(Note: should have also written $f(0) = 0$, but this did not count against anyone.) We have $|g(x)| \leqslant M$ if $x \approx 0$ (1 pt.). Thus $-Mf(x) \leqslant f(x)g(x) \leqslant Mf(x)$ if $x \approx 0$ (1 pt.). Since $\lim_{x\to 0} \pm Mf(x) = 0$, we have by the Squeeze Theorem (1 pt.) that $\lim_{x\to 0} f(x)g(x) = 0$.

Pick a continuous function which is zero at exactly two points, such a $x^2 - 1$. Let $\delta(irrational) = 1$ and $\delta(rational) = 0$, then $(x^2 - 1)\delta(x)$ is continuous exactly at the zeroes, (in this case ± 1.) (2 pts.)

Problem 3. Define sequential compactness. Show that if $S_1, \ldots, S_n$ are sequentially compact sets, then $S = S_1 \cup \cdots \cup S_n$ is also sequentially compact.

A set S is sequentially compact if for any sequence (x_n), there is a subsequence (x_{n_i}) and $x \in S$ so that $x_{n_i} \to x \in S$. (2 pts.)

Let (x_n) be a sequence in S. Since $S_1, \ldots, S_n$ is a finite collection, there is one, S_k, which contains a subsequence (x_{n_i}), (1.5 pts.) by sequential compactness, there is a subsequence of that which converges to a point $x \in S_k \subset S$. Thus (x_n) has a subsequences that converges to a point in S. (1.5 pts.)

Problem 4. Show that $e^x - \pi x = 0$ has at least two positive solutions.

$e^0 - \pi \cdot 0 = 1 > 0$, $e^1 - \pi \cdot 1 = e - \pi < 2.7 - 3.2 < 0$, $e^2 - 2\pi > (2.7)^2 - 6.4 > 0$ (3 pts.) By the Intermediate Value Theorem (1 pt.) there is $x_1 \in (0,1)$ and $x_2 \in (1,2)$ where $e^x - \pi x$ takes the value 0. (1 pt.)

Problem 5. Find the radius of convergence of

$$\sum_{k=2}^{\infty} \frac{(2k)!}{\ln(k)^2 (k!)^2} x^{2k}.$$

Since $x^{2n} \geqslant 0$, we have that

$$\sum_{k=2}^{\infty} \frac{(2k)!}{k^2 (k!)^2} x^{2k} \leqslant \sum_{k=2}^{\infty} \frac{(2k)!}{\ln(k)^2 (k!)^2} x^{2k} \leqslant \sum_{k=2}^{\infty} \frac{(2k)!}{(k!)^2} x^{2k}.$$

Since the left and right series have the same radius of convergence, so does the middle series. (1 pt.)

$$\frac{(2(k+1))!}{((k+1)!)^2} \Big/ \frac{(2k)!}{(k!)^2} = (2k+2)(2k+1)/(k+1)^2 \to 4$$

(2 pts.) so

$$\sum_{k=2}^{\infty} \frac{(2k)!}{\ln(k)^2 (k!)^2} x^{k}.$$

converges for $|x| < 1/4$ (1 pt.). Substituting x^2 for x, we have $|x^2| < 1/4$ of $|x| < 1/2$ (1 pt.)

Problem 6. State the definition of uniform continuity. Show that a function f defined on $[0, 1]$ is uniformly continuous if $(f(x_n))$ is a Cauchy sequence whenever (x_n) Cauchy sequence in $[0, 1]$.

Definition: $\forall \epsilon > 0$, $\exists \delta > 0$, so that for all c, x in the domain of f if $|x - c| < \delta$, then $|f(x) - f(c)| < \epsilon$. (2 pts.)

Let (x_n) be a Cauchy sequence with $x_n \to c$. Then $x_1, c, x_2, x_3, c, \ldots$ is also Cauchy since it still converges to c. (1 pt.) We have $f(x_1), f(c), f(x_2), f(c), f(x_3), \ldots$ is Casuchy by assumption which means $f(x_n) \to f(c)$ as $f(c)$ is a cluster point. This show that f is sequentially continuous, so continuous. (1 pt.) Since f is continuous on a compact interval, it is uniformly continuous (1 pt.)