MATH 341, EXAM #3 KEY

Problem 1. Give an example of a function which is differentiable for all real numbers but not twice differentiable. Give an example of a function which is defined for all real numbers but is differentiable only at 0.

$f(x) = x^{4/3}$ is an example for the first (2.5 points). Let $\delta(x)$ be the Dirichlet function, which is 1 if x is rational and 0 is x is irrational. $f(x) = x^2\delta(x)$ is an example for the second (2.5 pts) since

$$\lim_{x\to 0} \frac{x^2\delta(x) - 0}{x - 0} = \lim_{x\to 0} x\delta(x) = 0$$

while $x^2\delta(x)$ is still discontinuous at every other point, so cannot be differentiated.

Problem 2. State the Mean Value Theorem. Show that if $f(x)$ is differentiable and changes sign over some interval $[a, b]$, then there is a point $c \in [a, b]$ with $f'(c) \neq 0$.

The Mean Value Theorem states that if a function is continuous on $[a, b]$ and differentiable on (a, b) (1 pt.), then there is $c \in (a, b)$ so that

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

(1 pt.)

Without loss of generality, we may assume that for some $c, d \in [a, b]$, with $c < d$ that $f(c) < 0$ and $f(d) > 0$ (1 pt.). Since f is continuous on [a,b] and differentiable on (a, b), the same applies the the interval $[c, d]$ (1 pt.) By the MVT there is $e \in (c, d)$ so that

$$f'(e) = \frac{f(d) - f(c)}{d - c} > 0$$

(1 pt.)

Problem 3. Suppose f is a function which is twice differentiable on some interval I. Show that if $f(x) = 0$ at three points in I, then there is a point $c \in I$ with $f''(c) = 0$.

Pick three points $a, b, c \in I$ with $a < b < c$ so that $f(a) = f(b) = f(c) = 0$. Since f is twice differentiable on I, it is continuous and differentiable on each of $[a, b], [b, c]$ (1.5 pt.) Therefore by MVT/Rolle's, there are points $x \in (a, b)$ and $y \in (b, c)$ where $f'(x) = f'(y) = 0$ (1 pt.) We have that f' is differentiable, whence continuous, on $[x, y]$ (1.5 pt.), so by applying MVT/Rolle's a second time we can find $z \in (x, y)$ where $f''(z) = 0$. Thus there is a point $z \in I$ where $f''(z) = 0$. (1 pt.)

Problem 4. State Taylor's Remainder Theorem. Calculate $\cos(.1)$ to seven decimal places.

Let $f(x)$ on I be $(n+1)$ times differentiable and $a, x \in I$. Let $T_n(x)$ be the n-th Taylor polynomial for f based at a (1 pt.) Then there exists c between a and x so that

$$f(x) = T_n(x) + \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

(1 pt.).

For $\cos(0.1)$, consider T_5, the error in the Taylor estimate is at most

$$\sup_{[0,0.1]} \frac{f^{(6)}(c)}{720}(0.1-0)^6 < \frac{(0.1)^6}{720} < (0.1)^8/2$$

which gives at least 7 digits of precision (2 pts.)

Thus $\cos(0.1) \approx 1 - (0.1)^2/2 + (0.1)^4/24 \approx 0.99500417$ (1 pt.)

Problem 5. State the definition of Riemann integrability. Show that if $f(x)$ is Riemann integrable on $[a, b]$, then so is $[f(x)]^2$.

A bounded function on $[a, b]$ is Riemann integrable if for every $\epsilon > 0$ there exist $\delta > 0$ so that when $\mathcal{P} = \{x_0 < x_1 < \cdots < x_n\}$ is a partition with $|\mathcal{P}| < \delta$ (1 pt.), then

$$U_f(\mathcal{P}) - L_f(\mathcal{P}) = \sum_{i=0}^{n-1} \left(\sup_{[x_i, x_{i+1}]} f(x) - \inf_{[x_i, x_{i+1}]} f(x) \right) (x_{i+1} - x_i) < \epsilon$$

(1 pt.)

We have that $f^2 = |f|^2$. Since $\sup_{[a,b]} f^2(x) = (\sup_{[a,b]} f(x))^2$ when $f \geqslant 0$, and the same holds for inf, (1 pt.) we have that

$$U_{f^2}(\mathcal{P}) - L_{f^2}(\mathcal{P}) =$$

$$= \sum_{i=0}^{n-1} \left(\sup_{[x_i, x_{i+1}]} f^2(x) - \inf_{[x_i, x_{i+1}]} f^2(x) \right) (x_{i+1} - x_i)$$

$$\leqslant 2M \sum_{i=0}^{n-1} \left(\sup_{[x_i, x_{i+1}]} |f|(x) - \inf_{[x_i, x_{i+1}]} |f|(x) \right) (x_{i+1} - x_i) < 2M\epsilon$$

when $|\mathcal{P}| < \delta$ where M is the bound for $|f|$ (2 pts.)

Problem 6. Use only the definition of the Riemann integral to show that if $f(x) \geqslant 0$ is integrable on $[a, b]$, then $\int_a^b f(x)dx \geqslant 0$. Show that

$$\int_0^1 \frac{\sin(x)}{1+x^2} dx < \frac{\pi}{4}.$$

For any partition $\mathcal{P}$, we have that

$$L_f(\mathcal{P}) = \sum_{i=0}^{n-1} \left(\inf_{[x_i, x_{i+1}]} f(x) \right) (x_{i+1} - x_i) \geqslant 0$$

since each term is at least 0. (1.5 pts.) We have that $\int_a^b f(x)dx = \lim_{n\to\infty} L_f(\mathcal{P}_n) \geqslant 0$ for any sequence of partitions with $|\mathcal{P}_n| \to 0$ and $\mathcal{P}_{n+1} \leqslant \mathcal{P}_n$ for all n. (1.5 pts.)

We have that

$$\int_0^1 \frac{\sin(x)}{1+x^2} dx < \int_0^1 \frac{1}{1+x^2} = \arctan(1) - \arctan(0) = \frac{\pi}{4}.$$

Since $\sin(x) < 1$ on $[0, 1]$ (2 pts.)