MATH 341, ASSIGNMENT #1
DUE: WEDNESDAY, JANUARY 16, IN CLASS

Q1. Show that $1.3579\overline{579}$ is rational. Describe a procedure for showing that any real number which has a decimal expansion which eventually repeats is rational.

Q2. Show that $\sqrt{3}$ is irrational.

Q3. It is known that $\alpha = \sqrt{2} + \sqrt{3}$ is irrational.[1] Find a polynomial with integer coefficients so that $p(\alpha) = 0$.

A real number which is a root of a polynomial with integer coefficients is called an algebraic number. There are plenty of examples of irrational algebraic numbers. However, many real numbers, such as e, are not algebraic.

Q4. Let $A, B \subseteq X$. Draw pictures illustrating why $(A \cup B)^c = A^c \cap B^c$ and $(A \cap B)^c = A^c \cup B^c$.

Q5. Let $A_i \subset X$, $i \in I$ be a collection of subsets of a set X indexed by I. Give a proof of DeMorgan's Laws

$$\left(\bigcup_{i \in I} A_i\right)^c = \bigcap_{i \in I} (A_i)^c$$

and

$$\left(\bigcap_{i \in I} A_i\right)^c = \bigcup_{i \in I} (A_i)^c.$$

Q6. For $A, B, C \subseteq X$, show the following.

(1) $A \subseteq B$ if and only if $A \cup B = B$.
(2) $A \subseteq B$ if and only if $A \cap B = A$.
(3) $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$.

Q7. Let $f : X \to Y$ be a function. Show that $f(A \cup B) = f(A) \cup f(B)$. Does $f(A \cap B) = f(A) \cap f(B)$? If not, is there an inclusion?

[1] If you enjoy a challenge, can you show this?

Let $f : X \to Y$ be a function. For $B \subseteq Y$ we define the preimage (written as $f^{-1}(B)$)[2] of B under f to be

$$f^{-1}(B) = \{x \in X : f(x) \in B\}.$$

Q8. Show for $A, B \subseteq Y$ that $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$ and $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$.

Let X be a set. The characteristic function f_A of a subset $A \subseteq X$ is the function such that $f_A(x) = 1$ when $x \in A$ and $f_A(x) = 0$ when $x \notin A$

Q9. For $A, B \subseteq X$, can you express the characteristic functions of the following sets in terms of f_A and f_B?

(1) $A \cap B$
(2) A^c
(3) $A \cup B$

The Well Ordering Principle is the painfully obvious statement that any subset $S \subseteq \mathbb{N}$ of the natural numbers (which is not the empty set!!) contains a smallest element.

Q10. Show that the Well Ordering Principle implies that Proof by Mathematical Induction is logically valid.[3]

Q11. Let U be the collection whose elements are all possible sets. (This is quite an impressive collection!) Should U be considered a set?

We let $[n]$ be the set $\{1, \dots, n\}$ consisting of the first n natural numbers. The Pigeonhole Principle is the statement that for all $n \in \mathbb{N}$ a function $f : [n] \to [n]$ is injective if and only if it is surjective.

Challenge. Use induction to prove the Pigeonhole Principle.

[2] This is just notation, and should not be interpreted as f having an inverse!
[3] Challenge: show the converse.