MATH 341, QUIZ #3
FRIDAY, FEBRUARY 15

Question 0.0.1. Determine the radius of convergence of the power series

$$\sum_{n=1}^{\infty} n^2 \left(\frac{n}{n+1}\right)^{n^2} x^n.$$

You may use that

$$\lim_{n\to\infty} \left(1+\frac{a}{n}\right)^n = e^a.$$

Proof. Let $a_n = n^2 \left(\frac{n}{n+1}\right)^{n^2}$. We know that

$$\frac{1}{R} = \lim_{n\to\infty} |a_n|^{1/n},$$

if the limit exists, where R is the radius of convergence. We have that

$$|a_n|^{1/n} = (n^2)^{1/n} \left(\frac{n}{n+1}\right)^{n^2/n} = (n^{1/n})^2 \frac{1}{\left(\frac{n+1}{n}\right)^n}.$$

We use that $\frac{n+1}{n} = 1 + \frac{1}{n}$, the existence of the limits $n^{1/n} \to 1$ and $(1+1/n)^n \to e \neq 0$, and the limit laws to get that

$$\lim_{n\to\infty} |a_n|^{1/n} = \frac{\left(\lim_{n\to\infty} n^{1/n}\right)^2}{\lim_{n\to\infty}(1+\frac{1}{n})^n} = 1^2/e,$$

so $1/R = 1/e$ or $R = e$. $\square$