## ASSIGNMENT 1

**Q2. Show that $\sqrt{3}$ is irrational**[3 pts]

Method 1: if assuming $\sqrt{3}$ is a rational number, then it can be written as a reduced form $\sqrt{3} = \frac{p}{q}$,and $p, q$ are relatively prime. then $p^2 = 3\ q^2$ which means $3|p^2$. Since $p$ could be written as $p = 3m + r, r = 0, 1, 2$, then $p^2 = 9m^2 + 6mr + r^2, r^2 = 0, 1, 4$. so if $3|p^2$,then $3|r^2$ which means $r = 0$, thus $3|p$, then $\exists m, s.t.\ p = 3m$, then $9m^2 = p^2 = 3q^2 \ \Rightarrow 3m^2 = q^2$. as discussed before, we can get $3|q$, which means 3 divides $p, q$, which is contradictive with assumption $p, q$ are relatively prime. Thus, $\sqrt{3}$ is not rational.

Method 2: if assuming $\sqrt{3}$ is a rational number, then it can be written as a reduced form $\sqrt{3} = \frac{p}{q}$,and $p, q$ are relatively prime and $p^2 = 3\ q^2$. Since $p, q \in N$ means $p, q$ has to be odd or even. If $q$ is even then $\exists m \in N, q = 2m, \Rightarrow p^2 = 3q^2 = 3 * 4 * m^2, \Rightarrow 4|p^2 \ \Rightarrow 2|p$ which means $p$ is also even. which is contradictive with assumption $p, q$ are relatively prime. Same idea can prove $p$ is not even, thus $p, q$ are odd. Then $\exists m, n,\ s.t.\ p = 2m + 1,\ q = 2n + 1$. Then $p^2 = 4n^2 + 4n + 1 = 3 * (4m^2 + 4m + 1) = 12m^2 + 12m + 3 = 3q^2, \Rightarrow 2n^2 + 2n = 6m^2 + 6m + 1$. which is impossible, since left side is even, right side is odd, they can not be identity. Thus, $\sqrt{3}$ is not rational.

**Q6. For A, B, C $\subset$ X, show the following.**
(1) $A \subset B$ if and only if $A \cup B = B$.
(2) $A \subset B$ if and only if $A \cap B = B$.
(3) $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$.

1)[1 pts] Show that $A \cup B = B$ if and only if $A \subset B$.
($\Rightarrow$) $\forall x \in A$, since $A \subset A \cup B$,and $A \cup B = B$, then $x \in B$, thus $A \subset B$.
($\Leftarrow$) if $A \subset B$, then $A \cup B \subset B \cup B = B$. Combined with $B \subset A \cup B$, can get $A \cup B = B$.

b)[1 pts] Show that $A \cap B = A$ if and only if $A \subset B$.
($\Rightarrow$) $\forall x \in A$, since $A = A \cap B$, $x \in A \cap B$. since $A \cap B \subset B$, thus $x \in B$, which means $A \subset B$.
($\Leftarrow$) if $A \subset B$, then $A = A \cap A \subset A \cap B$. Combined with $A \cap B \subset A$, can get $A \cap B = B$.

c)[1 pts] Show that $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$.
1),$(A \cup B) \cap C \supset (A \cap C) \cup (B \cap C)$ : $A \cap C \subset (A \cup B) \cap C$ and $B \cap C \subset (A \cup B) \cap C$, thus $(A \cup B) \cap C \supset (A \cap C) \cup (B \cap C)$.
2),$(A \cup B) \cap C \subset (A \cap C) \cup (B \cap C)$ : $\forall x \in (A \cup B) \cap C$ means $x \in C\ and\ x \in A\ or\ B$. if $x \in A, then\ x \in A \cap C, then\ x \in (A \cap C) \cup (B \cap C), if\ x \in B, then\ x \in B \cap C, then\ x \in (A \cap C) \cup (B \cap C), thus\ (A \cup B) \cap C \subset (A \cap C) \cup (B \cap C)$.

**Q8. Show for A, B $\subset Y$ that $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$ and $f^{-1}(A \cup B) = f^{-1}(A) \cap f^{-1}(B)$.**

a)[2 pts] $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$.

1),$f^{-1}(A \cup B) \subset f^{-1}(A) \cup f^{-1}(B)$: $\forall x \in f^{-1}(A \cup B)$ means $f(x) \in A \cup B$, then $x \in f^{-1}(A) \cup f^{-1}(B)$.
2),$f^{-1}(A) \cup f^{-1}(B) \subset f^{-1}(A \cup B)$: $f^{-1}(A) \subset f^{-1}(A \cup B)$ and $f^{-1}(B) \subset f^{-1}(A \sup B)$, then $f^{-1}(A) \cup f^{-1}(B) \subset f^{-1}(A \cup B)$.

b)[2 pts] $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$.
1), $f^{-1}(A \cap B) \subset f^{-1}(A) \cap f^{-1}(B)$: $f^{-1}(A \cap B) \subset f^{-1}(A)$ and $f^{-1}(A \cap B) \subset f^{-1}(B)$, then $f^{-1}(A \cap B) \subset f^{-1}(A) \cap f^{-1}(B)$.
2), $f^{-1}(A) \cap f^{-1}(B) \subset f^{-1}(A \cap B)$: $\forall x \in f^{-1}(A) \cap f^{-1}(B)$ means $f(x) \in A$ and $f(x) \in B$, then $f(x) \in f^{-1}(A \cap B)$.

**Challenge. Use induction to prove the Pigeonhole Principle.**

Prove by induction. For $n = 1$, it is obviously. Assume the statement holds for $n = k$, for $n = k + 1$,
case 1: if $f(k + 1) = k + 1$, then by the assumption that $f : [k] \to [k]$ is injective if and only if it is surjective, shows $f$ is injective if and only if $f$ is surjective.
case 2: if $f(k + 1) = a, f(b) = k + 1$, for $a \neq k + 1$. Define a function $g$ as $g(k + 1) = k + 1, g(b) = a, g(s) = f(s)$ for $s \neq k + 1, b$, then $g$ satisfies case 1, shows that $g$ is injective if and only if $g$ is surjective. By the definition of $g$, we can see, $g$ is injective if and only if $f$ is injective, $g$ is surjective if and only if $f$ is surjective, which proves the statement for $n = k + 1$.
Complete the proof by principle of induction.

**Completeness: [0/-1 pts].**