ASSIGNMENT 2

[3 pts] Q3: If $a_n = \sum\limits_{i=1}^{n} \frac{1}{i}$, use induction to show that $a_{2^n} \geq \frac{n}{2}$.

Proof. For $n = 1$, $a_1 = 1 > \frac{1}{2}$. Assume the statement holds for $n = k$ such that $a_{2^k} \geq \frac{k}{2}$, then for $n = k+1$,

$$a_{2^{k+1}} = a_{2^k} + \sum_{i=2^k+1}^{2^{k+1}} \frac{1}{i} \geq \frac{k}{2} + \sum_{i=2^k+1}^{2^{k+1}} \frac{1}{2^{k+1}} \geq \frac{k+1}{2}.$$

By method of induction, complete the proof. □

[3 pts] Q8: Let (a_n) be a sequence so that $a_n \in \{0, ..., 9\}$ for each n. Express using quantifiers the statement that $a_1.a_2a_3a_4...$ is the decimal expansion of a rational number.

Proof. For a given sequence (a_n), if $\exists N, k \in \mathbb{N}^+$, such that $\forall n > N$, $a_{n+k} = a_n$, it is a decimal expansion of a rational number. □

[4 pts] Q10: Consider the sequence $\{a_n\}$, where

$$a_n = 1 + \frac{1}{1 \cdot 3} + ... + \frac{1}{1 \cdot 3 \cdot ... \cdot 2n - 1}.$$

Decide whether a_n is bounded above or not, and prove your answer is correct.

Proof. It is bounded above. As for any $n \geq 1$,

$$a_n = 1 + \frac{1}{1 \cdot 3} + ... + \frac{1}{1 \cdot 3 \cdot ... \cdot 2n - 1} \leq \sum_{k=1}^{n} \frac{1}{2^{k-1}} < 2.$$

□

Challenge: Find all sequences (a_n) of non-negative real numbers so that $\sum\limits_{i=1}^{n} a_i^3 = \left(\sum\limits_{i=1}^{n} a_i\right)^2$ for all n.

Proof. Denote $b_n = \max\limits_{i<n} a_i$. Then $a_n = 0$ or $b_n + 1$. Prove by induction. For $n = 1$, since $a_1^3 = a_1^2$, $a_1 = 0$ or 1. Assume the statement holds for $n = k$, then for $n = k+1$, assume $b_{k+1} = m$, then by the statement, $\sum\limits_{i=1}^{k} a_i = \sum\limits_{j=0}^{m} j = \frac{m(m+1)}{2}$. Then

$$\begin{aligned}
\sum_{i=1}^{k+1} a_i^3 = \left(\sum_{i=1}^{k+1} a_i\right)^2 \quad &\Rightarrow \quad \sum_{i=1}^{k} a_i^3 + a_{k+1}^3 = \left(\sum_{i=1}^{k} a_i\right)^2 + a_{k+1}^2 + 2a_{k+1}\left(\sum_{i=1}^{k} a_i\right) \\
&\Rightarrow \quad a_{k+1}^3 = a_{k+1}^2 + 2a_{k+1}\left(\sum_{i=1}^{k} a_i\right) \\
&\Rightarrow \quad a_{k+1}\left(a_{k+1}^2 - a_{k+1} - m(m+1)\right) = 0 \\
&\Rightarrow \quad a_{k+1}\left(a_{k+1} + m\right)\left(a_{k+1} - (m+1)\right) = 0.
\end{aligned}$$

Since a_{k+1} is a non-negative real number, so $a_{k+1} = 0$ or $m+1$, which completes the proof. □

Completeness: [0/-1 pts].