ASSIGNMENT 3

Q1 [3 pts]: If $a \approx_\epsilon b$, and $|a| \leq K$, $|b| \leq K$, give an estimate for the accuracy of the approximation $a^n \approx b^n$, where n is a positive integer.

Proof. $|a^n - b^n| = \left|(a-b)\sum_{k=0}^{n-1} b^k a^{n-1-k}\right| \leq |a-b| \sum_{k=0}^{n-1} \left|b^k a^{n-1-k}\right| \leq \epsilon n K^{n-1}$. So $a^n \approx_{\epsilon n K^{n-1}} b^n$. $\square$

Q4 [4 pts]:
(a), Prove the sequence $a_n = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}$ has a limit.
(b), Criticize the following "proof" that its limit is 0: Give $\epsilon > 0$, then for $i = 1, 2, 3...$, we have $\frac{1}{n+i} < \epsilon$, if $\frac{1}{n} < \epsilon$, i.e. if $n > \frac{1}{\epsilon}$. Adding up these inequalities for $i = 1, ..., n$ gives $0 < a_n < n\epsilon$, for $n > \frac{1}{\epsilon}$; therefore $a_n \approx_{n\epsilon} 0$, for $n >> 1$. By the definition of limit and the $K - \epsilon$ principle, $\lim a_n = 0$.

Proof. a), since $a_{n+1} - a_n = \frac{1}{2n+2} + \frac{1}{2n+1} - \frac{1}{n+1} > 0$, for all $n > 0$, sequence a_n is increasing. Since $a_n < \sum_{k=1}^{n} \frac{1}{n+1} \leq 1$, then a_n is an increasing sequence which a upper bound, so it has a limit.
b), $K - \epsilon$ principle can not be used, as K dependents on n. $\square$

Q8 [3 pts]: Prove or provide a counterexample for each of the following statements:
(1), if (a_n) and (b_n) are bounded above, so is $(a_n b_n)$.
(2), if (a_n) and (b_n) are both divergent, then $(a_n + b_n)$ is divergent.
(3), if (a_n) and $a_n b_n$ are both convergent, then so is (b_n).
(4), if (a_n) is convergent, then so is (a_n^2).
(5), if (a_n^2) is convergent, then so is (a_n).
(6), if $a_n < b_n$ both converge, then $\lim_{n\to\infty} a_n < \lim_{n\to\infty} b_n$.

Proof. (1), False. For example $a_n = -n = b_n$.
(2), False. For example $a_n = n, b_n = -n$.
(3), False. For example $a_n = 0, b_n = n$.
(4), True. Assume $\lim_{n\to\infty} a_n = L$, then $\forall \epsilon > 0$, and $\epsilon < L, \exists N > 0, \forall n > N, |a_n - L| < \epsilon$. Then $|a_n^2 - L^2| = |a_n - L||a_n + L| < 3L\epsilon$. By $K - \epsilon$ principle, $\lim_{n\to\infty} a_n^2 = L^2$.
(5), False. For example $a_n = (-1)^n$.
(6), False. For example $a_n = -\frac{1}{n}, b_n = \frac{1}{n}$. $\square$

Challenge: Let $a_1 = \sqrt{2}$ and define a_n recursively by $a_{n+1} = \sqrt{2 + a_n}$. Show the limit exists and compute it.

Proof. First, a_n is an increasing sequence. Prove by induction. For $n = 2, a_2 = \sqrt{2+2} > \sqrt{2}$. Assume it holds for $n = k$, then for $n = k+1$, $a_{k+1} - a_k = \sqrt{2 + a_k} - \sqrt{2 + a_{k-1}} > 0$, which prove a_n is increasing. Second, a_n is bounded above by 2. Prove by induction. For $n = 1, a_1 = \sqrt{2} < 2$. Assume it holds for $n = k$, then for $n = k+1$, $a_{k+1} = \sqrt{2 + a_k} \leq \sqrt{2+2} = 2$. Since a_n is an increasing sequence with a upper bound, a_n has a limit. Assume $\lim_{n\to\infty} a_n = L$, then $L = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \sqrt{2 + a_n} = \sqrt{2 + L}$, gives $L = 2$. $\square$

Completeness: [0/-1 pts].