ASSIGNMENT 4

Q2 [3 pts]: Show that if $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| < 1$, then $\lim_{n\to\infty} a_n = 0$.

Proof. By $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| < 1$, $\exists N$ such that $\forall n > N$, $\left|\frac{a_{n+1}}{a_n}\right| \le r < 1$. Assume $|a_N| = C > 0$. Note that if $C = 0$, then $a_n = 0$ for $n > N$. $\forall \epsilon > 0$, choose N_1 such that $r^{N_1-1-N} < \frac{\epsilon}{C}$, then $\forall n > N_1$,

$$|a_n| \quad = \quad |a_N|\,\Pi_{k=N}^{n-1}\left|\frac{a_{k+1}}{a_k}\right| \le |a_N| r^{n-1-N} \le |a_N| r^{N_1-1-N} < \epsilon.$$

□

Q5 [3 pts]: Show that $\lim_{n\to\infty}\sum_{i=n}^{2n}\frac{1}{i} = \ln 2$. Use the method from Example 5.2C on p.66.

Proof. For $n \ge 2$,

$$\sum_{i=n}^{2n}\int_i^{i+1}\frac{1}{x}\,dx \le \sum_{i=n}^{2n}\int_i^{i+1}\frac{1}{i} \le \sum_{i=n}^{2n}\int_i^{i+1}\frac{1}{x-1}\,dx$$

$$\Rightarrow \int_n^{2n+1}\frac{1}{x}\,dx \le \sum_{i=n}^{2n}\frac{1}{i} \le \int_n^{2n+1}\frac{1}{x-1}\,dx$$

as $\lim_{n\to\infty}\int_n^{2n+1}\frac{1}{x}\,dx = \lim_{n\to\infty}\int_n^{2n+1}\frac{1}{x-1}\,dx = \ln 2$. Then completes the proof by squeeze Theorem. □

Q8 [4 pts]: Suppose a sequence $\{a_n\}$ has this property: there exist constants C and K, with $0 < K < 1$, such that $|a_n - a_{n+1}| < CK^n$, for $n \gg 1$. Prove that $\{a_n\}$ is a Cauchy sequence.

Proof. By $|a_n - a_{n+1}| < CK^n$, for $n \gg 1$, $\exists N > 0$ such that $\forall n > N$, $|a_n - a_{n+1}| < CK^n$. $\forall \epsilon > 0$, choose N_1 such that $\frac{C}{1-K}K^{N_1} < \epsilon$ and $N_1 > N$, then $\forall m, n > N_1$, Wlog, assume $m > n$, we have

$$|a_m - a_n| \le \sum_{j=n}^{m-1}|a_j - a_{j+1}| \le \sum_{j=n}^{m-1} CK^j = CK^n\frac{1-K^{m-n}}{1-K} \le CK^{N_1}\frac{1}{1-K} < \epsilon.$$

□

Challenge: Show the set of cluster points of $\sin(n)$ is exactly $[-1,1]$.

Proof. Denote $\lfloor a \rfloor$ be the largest integer less than a. Then the question is equivalent to prove $\left\{\frac{n}{2\pi} - \lfloor\frac{n}{2\pi}\rfloor \,\middle|\, n \in \mathbb{N}\right\}$ is dense in $[0,1]$. First let's construct a sequence in the following way: $a_1 = \frac{1}{2\pi}, n_1 = 1$. As $\frac{1}{2\pi}$ is irrational, there exists integer m_1 such that $\frac{1}{2\pi}(m_1 - 1) < 1 < \frac{m_1}{2\pi}$, then $\frac{m_1}{2\pi} - 1 = \frac{m_1}{2\pi} - \lfloor\frac{m_1}{2\pi}\rfloor < a_1$, denote $a_2 = \frac{m_1}{2\pi} - \lfloor\frac{m_1}{2\pi}\rfloor$ and $n_2 = n_1 m_1$, repeat above steps, we can generate a strictly decreasing sequence $\{a_i\}_{i=1}^{\infty}$.

Prove the question by contradiction. If assume $\left\{\frac{n}{2\pi} - \lfloor\frac{n}{2\pi}\rfloor \,\middle|\, n \in \mathbb{N}\right\}$ is not dense in $[0,1]$, then there exists a, b such that $0 < a < b < 1$, such that $(a,b) \cap \left\{\frac{n}{2\pi} - \lfloor\frac{n}{2\pi}\rfloor \,\middle|\, n \in \mathbb{N}\right\} = \emptyset$ and $\exists N \in \mathbb{N}$ that $a = \frac{N}{2\pi} - \lfloor\frac{N}{2\pi}\rfloor$. As the sequence defined in first part is strictly decreasing to zero, $\exists a_k = \frac{n_k}{2\pi} - \lfloor\frac{n_k}{2\pi}\rfloor < \frac{b-a}{2}$, then $a < a + a_k = \frac{N+n_k}{2\pi} - \lfloor\frac{N+n_k}{2\pi}\rfloor < b$ which contracts to the assumption. □

Completeness: [0/-1 pts].