ASSIGNMENT 5

Q4 [3 pts]: (a) Prove: $\sum a_n$ absolutely convergent $\Rightarrow \sum a_n^2$ convergent. (b) Show that "absolutely" cannot be dropped in part (a).

Proof. (a), As $\sum a_n$ is absolutely convergent, then $\exists N > 0$, such that $|a_n| < 1$ for $n > N$, then $\sum\limits_{n>N} a_n^2 < \sum\limits_{n>N} |a_n| < \infty$, which proves $\sum a_n^2$ is also convergent.

(b), For example, $\left\{a_n = (-1)^n \frac{1}{\sqrt{n}}\right\}$. □

Q5[3 pts]: Test each of the following series for convergence.

Proof. (d) $\sum\limits_{n=0}^{\infty} \frac{(n!)^2}{(2n)!}$. By ratio test $\lim\limits_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty} \frac{(n+1)^2}{(2n+2)(2n+1)} = \frac{1}{4}$, so it is absolutely convergent.

(e) $\sum\limits_{n=1}^{\infty} \left(\frac{n+1}{2n+1}\right)^n$. By the n-th root test $\lim\limits_{n\to\infty} |a_n|^{1/n} = \lim\limits_{n\to\infty} \frac{n+1}{2n+1} = \frac{1}{2}$, so it is absolutely convergent.

(h) $\sum\limits_{n=1}^{\infty} \frac{2^n n!}{n^n}$. By ratio test $\lim\limits_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty} 2(1 - \frac{1}{1+n})^n = \frac{2}{e}$, so it is absolutely convergent.

(j) $\sum\limits_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$. By integral test define $f(x) = \frac{1}{x(\ln x)^p}$, then for $p \geq 0$ and $x \geq 3$, $f(x) \geq 0$ and decreasing. Then $\int_3^\infty f(x)\,dx = \frac{1}{1-p}(\ln x)^{1-p}\Big|_3^\infty$ which is finite for $p > 1$ and infinite for $p < 1$. So the series convergent for $p > 1$ and divergent for $0 \leq p < 1$. For $p = 1$, define $f(x) = \frac{1}{x\ln x}$ and $\int_2^\infty \frac{1}{x\ln x}\,dx = \ln\ln x\Big|_2^\infty$ is infinity, by integral test, the series divergent for $p = 1$. For $p < 0$, since $\frac{1}{n(\ln n)^p} > \frac{1}{n}$, the series is divergent. □

Q7[4 pts]: Find the radius of convergence for the following power series.

Proof. (a), $\sum\limits_{n=1}^{\infty} \frac{x^n}{2^n\sqrt{n}}$, by ratio test, $\lim\limits_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty} \frac{|x|}{2}\sqrt{\frac{n}{n+1}} = \frac{|x|}{2}$, so radius is $R = 2$.

(b), $\sum\limits_{n=1}^{\infty} \frac{(n!)^2}{(2n)!}x^n$, by ratio test, $\lim\limits_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty} \frac{(n+1)^2}{(2n+2)(2n+1)}|x| = \frac{|x|}{4}$, so radius is $R = 4$.

(c), $\sum\limits_{n=1}^{\infty} \frac{x^n}{\sqrt[n]{n}}$, by n-th root test, $\lim\limits_{n\to\infty} |a_n|^{1/n} = \lim\limits_{n\to\infty} \frac{|x|}{n^{1/n^2}} = |x|$, so radius is $R = 1$.

(d), $\sum\limits_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{4^n (n!)^2}$, by ratio test, $\lim\limits_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty} \frac{x^2}{4(n+1)^2} = 0$, so radius is $R = \infty$.

(e), $\sum\limits_{n=1}^{\infty} \left(\frac{n+2}{n}\right)^n x^n$, by n-th root test, $\lim\limits_{n\to\infty} |a_n|^{1/n} = \lim\limits_{n\to\infty} \frac{n+2}{n}|x| = |x|$, so radius is $R = 1$.

(f), $\sum\limits_{n=2}^{\infty} \frac{x^n}{(\ln n)^n}$, by n-th root test, $\lim\limits_{n\to\infty} |a_n|^{1/n} = \lim\limits_{n\to\infty} \frac{|x|}{\ln n} = 0$, so radius is $R = \infty$.

(g), $\sum\limits_{n=1}^{\infty} \frac{n^n x^n}{n!}$, by ratio test, $\lim\limits_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty} \left(1 + \frac{1}{n}\right)^n |x| = e|x|$, so radius is $R = \frac{1}{e}$.

(h), $\sum\limits_{n=1}^{\infty} \frac{n! x^n}{1\cdot 3\cdot 5\cdots(2n-1)}$, by ratio test, $\lim\limits_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty} \frac{n+1}{2n+3}|x| = \frac{|x|}{2}$, so radius is $R = 2$. □

Challenge: For each series in Exercise 8.1/1 above, determine whether it converges at the endpoints $\pm R$ of the interval of convergence, if it is easy to do so.

Proof. (a), At $x = 2$, $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ is divergent. At $x = -2$, $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ is convergent.

(b), At $x = 4$, $\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!} 4^n = \sum_{n=1}^{\infty} \frac{(2n)\cdot(2n-2)\cdots 2}{(2n-1)\cdot(2n-3)\cdots 1} > \sum_{n=1} 1$ is divergent.
At $x = -4$, $\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!} 4^n (-1)^n$ by Raabe's test, it is divergent.

(c), At $x = 1$, $\sum_{n=1}^{\infty} \frac{1}{\sqrt[n]{n}}$ is divergent. At $x = -1$, $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt[n]{n}}$ is divergent.

(e), At $x = 1$, $\sum_{n=1}^{\infty} \left(\frac{n+2}{n}\right)^n$ is divergent. At $x = -1$, $\sum_{n=1}^{\infty} \left(\frac{n+2}{n}\right)^n (-1)^n$ is divergent.

(g), At $x = \frac{1}{e}$, $\sum_{n=1}^{\infty} \frac{(n/e)^n}{n!}$, by Raabe's test, $\rho_n = \frac{1}{2}$, so it is divergent.

(h), Same with (b). □

Challenge: Determine, with proof, the radius of convergence of $\sum (\sin n) x^n$.

Proof. For $|x| < 1$, then $\sum |(\sin n) x^n| \leq \sum |x^n|$ is convergent. For $|x| = 1$, as discussed in the Challenge problem of HW 4, $\{\sin n\}$ is dense in $[-1, 1]$, then define subsequece $a_m = |\sin(n)|$, for any n such that $\frac{1}{m+1} < |\sin n| < \frac{1}{m}$, then $\sum |\sin n| > \sum \frac{1}{m} = \infty$. So the radius is 1. □

Completeness: [0/-1 pts].