ASSIGNMENT 6

Q3[3 pts]: Given an example of a function which is periodic, non-constant, and yet has no minimal period.

Proof. Define f as

$$f(x) = \begin{cases} 1, & x \text{ is rational}, \\ 0, & x \text{ is irrational}, \end{cases}$$

□

Q4[3 pts]: Show: a periodic increasing function is constant.

Proof. Denote this function by f and assume Λ is the period. For any x, y such that $x < y$, as f is increasing, $f(x) \leq f(y)$; Since f is periodic, $\exists N \in \mathbb{Z}$ such that $y < x + N\Lambda$ and $f(x) = f(x + N\Lambda)$, then $f(x) \leq f(y) \leq f(x + N\Lambda) = f(x)$ which proves $f(x) = f(y)$. □

Q6[4 pts]: If you approximate $\ln(1+x)$ by using the first three terms of the series $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$, estimate the error when $0 \leq x \leq 0.1$.

Proof. Define function sequence: $f_n(x) = \sum_{k=1}^{n} (-1)^{k-1} \frac{x^k}{k}$, then follow the discussion of Cauchy's test in section 7.6, we can get the same estimate with (15) as

$$|f_n(x) - \ln(1+x)| \leq \frac{x^{n+1}}{n+1},$$

by choosing $n = 3$, $|f_3(x) - \ln(1+x)| \leq \frac{10^{-4}}{4}$. □

Challenge: Prove that a function which is locally constant on [0,1) is actually constant on [0,1).

Proof. Define S as $S = \sup\{a < 1 : f(x) \text{ is constant on } [0, a)\}$, if $f(x)$ is not constant on [0,1), then $S < 1$. However, by the definition of S, $\forall \delta > 0$, $\exists x$ where $0 < x - S < \delta$ such that $f(x) \neq f(S - \frac{\delta}{2})$ which is contradict to $f(x)$ is locally constant. □

Completeness: [0/-1 pts].