ASSIGNMENT 7

Q2[3 pts]:

a) Prove that $\lim_{x\to 0^+} \frac{\sin x}{x} = 1$ **by first establishing the inequalities**

$$\cos x \le \frac{\sin x}{x} \le \frac{1}{\cos x}, \textbf{for } x \approx 0^+.$$

b) Deduce that $\lim_{x\to 0} \frac{\sin x}{x} = 1$ **by using one of the previous exercises.**

Proof. a), By comparing the areas, it is easy to get that for x close to 0^+ in positive direction,

$$\frac{1}{2}\sin x \cos x \le \frac{x}{2\pi}\pi \le \frac{1}{2}\tan x = \frac{1}{2}\frac{\sin x}{\cos x},$$

then prove that

$$\cos x \le \frac{\sin x}{x} \le \frac{1}{\cos x},$$

b), By squeeze theorem, $\lim_{x\to 0^+} \frac{\sin x}{x} = 1$, since both of x and $\sin x$ are even functions, so $\lim_{x\to 0} \frac{\sin x}{x} = \lim_{x\to 0^+} \frac{\sin x}{x} = 1$. □

Q6[3 pts]: Exercise 12.3.1 on P.181.

Proof. Assume $f(x)$ is a IVP and strictly decreasing function in (a,b). Then for any $\epsilon > 0$ and $x_0 \in (a, b)$, $\exists x_1 < x_0 < x_2$ such that $f(x_1) = f(x_0) + \epsilon, f(x_2) = f(x_0) - \epsilon$ by IVP and strictly decreasing property. Then for any $x \in (x_1, x_2)$, $f(x_1) > f(x) > f(x_2)$ which shows $|f(x) - f(x_0)| < \epsilon$ then completes the proof. □

Q9[4 pts]: Exercise 13.5.6 on P.193.

Proof. a), for $\forall \epsilon > 0$, let $\delta = \frac{\epsilon}{K}$, then for any $x_0 \in I$ and any $x \in [-\delta + x_0, x_0 + \delta] \cap I$, from the condition, we have

$$|f(x) - f(x_0)| \le K|x - x_0| \le \epsilon,$$

which shows it is uniformly continuous.

b), 1, The slope of $\sqrt{x}$ is not bounded at $x = 0$.

2, For $\forall \epsilon > 0$, let $\delta < \epsilon^2$, then for any $x_1, x_2 \in [0, 1]$ such that $|x_1 - x_2| < \delta$,

$$|\sqrt{x_1} - \sqrt{x_2}|^2 \le |\sqrt{x_1} + \sqrt{x_2}|\,|\sqrt{x_1} - \sqrt{x_2}| \le |x_1 - x_2| < \delta,$$

then $\left|\sqrt{x_1} - \sqrt{x_2}\right| \le \sqrt{\delta} < \epsilon$. □

Danish ham sandwich: A Danish open-faced ham sandwich consists of a thin slice of rye bread and a thin slice of ham. Both have irregular shapes, but assume they have uniform thickness. Prove it possible with a single vertical knife-cut to divide any ham sandwich in two so that each person gets half the ham and half the bread.

Proof. First, assume the shapes of ham and bread is not too irregular so that any single cut only divides ham and bread into two parts.

Denote θ be the angle of any cut, then for any θ, at least there is a way to cut bread to equal half. Proof: Denote the size of bread is S. As the bread is bounded, there exists a square box with width b contain it. For any $\theta \in [0, \pi]$, denote the part on the left of knife be A, the one on the right side be B, and their size be S_A, S_B, respectively. Denote $f = S_A - S_B$ be the difference. For any coordinate system, denote x be the intersection of cut and x-axis, then f is a continuous function of x. Since $\forall \epsilon > 0$, let $\delta < \frac{\epsilon}{2b}$, for any x_1, x_2 that $|x_1 - x_2| < \delta$,

$$|f(x_1) - f(x_2)| \leq |S_A(x_1) - S_A(x_2)| + |S_B(x_1) - S_B(x_2)| \leq 2\delta b < \epsilon.$$

Move the knife from the most left to the most right, f changes from $-S$ to S, then by IVP, there is a point for $f = 0$ which half divide the bread.

There is at least a cut both divides bread and ham to equal half. Proof: Based on above proof, for any $\theta \in [0, \pi]$, consider the cut that equal divides bread. Denote A, B be the left part and right part of ham divided by the cut. S_A, S_B, f are the size of A, B and difference, respectively. Then f is a continuous function of θ. Since the ham is bounded, then there exists a circle with radius R contains it. For any $\epsilon > 0$, let $\delta < \frac{\epsilon}{R^2}$, for any θ_1, θ_2 that $|\theta_1 - \theta_2| < \delta$,

$$|f(\theta_1) - f(\theta_2)| \leq |S_A(\theta_1) - S_A(\theta_2)| + |S_B(\theta_1) - S_B(\theta_2)| \leq 2\frac{\delta}{2\pi}\pi R^2 < \epsilon.$$

Without loss of generosity, assume $f(0) < 0$, then $\theta = \pi$ just switch A, B, so $f(\pi) > 0$, by IVP, there is a point for $f = 0$.

$\square$

Completeness: [0/-1 pts].