ASSIGNMENT 8

Q3[3 pts] (Exercise 14.3.3 on p. 207.): Let $f(x)$ be a polynomial of degree n.
a) What is the maximum number of critical points $f(x)$ can have? For each n, give an example, with proof, of a polynomial having the maximum number for that n.
b) Answer the same question with "maximum" replaced by "minimum".

Proof. a) For $n > 1$, at most, $f(x)$ can have $n-1$ critical points. For example, for each n, define $f(x) = (x-1)...(x-n)$; As $f(i) = 0$ for $i = 1, 2, ..., n$, by Rolle's rule, $\exists \xi_i \in [i, i+1]$, for $i = 1, ..., n-1$, such that $f'(\xi_i) = 0$ which completes the proof.
b) If n is odd, then the minimum number of critical points is 0; If n is even, then the minimum number is 1. For example, for $n = 2m+1$, define $f(x) = x^{2m+1}+x$, then $f'(x) = (2m+1)x^{2m}+1 > 0$ which shows that it has no critical points. For $n = 2m$, then for $x \to +\infty$ and $x \to -\infty$, $f(x)$ either tends to $+\infty$ or $-\infty$. Then there exists points x_1, x_2 such that $f(x_1) = f(x_2)$, by mean value theory, $\exists \xi \in [x_1, x_2]$ such that $f'(\xi) = 0$. So at least $f(x)$ has one critical points. For example $f(x) = x^{2m}$. □

Q6[3 pts](Exercise 15.2.1):
a) Prove that if $f'(x)$ is bounded on a finite interval I, then $f(x)$ is bounded on I. Is it true if "finite" is omitted?
b) Show that the converse is false.

Proof. a) As $f(x)$ is differentiable on I, at least exist $a \in I$ such that $f(a)$ is bounded. Then by mean value theory, for any $x \in I$, exists $c_x \in I$ such that $|f(x)| = |f(a) + f'(c_x)(x-a)| \leq |f(a)| + |f'(c_x)||x-a|$, as both $f'(x)$ and I is finite, so $f(x)$ is bounded in I. If the "finite" is omitted, then it is not true. For example, $f(x) = x$ and $I = \mathbb{R}$.
b) For example $f(x) = \sqrt{x}$, I=(0,1]. □

Q8[4 pts](Exercise 16.2.2): Assume $f(x)$ is differentiable on I; prove that if $f(x)$ is geometrically convex, then it is convex.

Proof. For any $a < b$, by definition of geometrical convexity, $\frac{f(x)-f(a)}{x-a} \leq \frac{f(b)-f(a)}{b-a}$ for any $a < x < b$ in I. Then

$$\frac{f(x)-f(a)}{x-a} \leq \frac{f(b)-f(x)}{b-a} - \frac{f(x)-f(a)}{b-a} \Rightarrow \frac{f(x)-f(a)}{x-a} \leq \frac{f(b)-f(x)}{b-x}.$$

By letting $x \to a^+$, $f'(a) \leq \frac{f(b)-f(a)}{b-a}$; by letting $x \to b^-$, $\frac{f(b)-f(a)}{b-a} \leq f'(b)$. Then $f'(a) \leq f'(b)$ shows $f'(x)$ is increasing, by first derivative test for convexity, $f(x)$ is convex.

□

Challenge: (Exercise 15.3.2): Let f and g be continuous on $[a, b]$ and differentiable on (a, b). Then there is a point $c \in (a, b)$ such that

$$f'(c)[g(b) - g(a)] = g'(c)[f(b) - f(a)]. \quad (0.1)$$ a

a) Deduce the two Mean-Value Theorems in the text (Theorem 15.1 and 15.3) from this formulation.
b) Prove (0.1) by applying the Mean-Value theorem to

$$F(t) = f(t)[g(b) - g(a)] - g(t)[f(b) - f(a)].$$

Proof. (a), Let $g(x) = x$, then $g'(x) = 1$, plug it back to (0.1) prove Theorem 15.1. Assume $g'(t) \neq 0$ on (a, b), and $g(b) \neq g(a)$, then

$$f'(c)[g(b) - g(a)] = g'(c)[f(b) - f(a)] \Rightarrow \frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

(b), $F(a) = f(a)g(b) - f(a)g(a) - g(a)f(b) + g(a)f(a) = f(a)g(b) - g(a)f(b)$ and $F(b) = f(b)g(b) - f(b)g(a) - g(b)f(b) + g(b)f(a) = g(b)f(a) - f(b)g(a)$. Then $F(a) = F(b)$, by Rolle's rule, $\exists c \in (a, b)$ such that $F(c) = f(c)[g(b) - g(a)] - g(c)[f(b) - f(a)] = 0$. □

Challenge: (Exercise 16.2.4):
a) Let $f(x)$ be geometrically concave on I, and $a, b \in I$. Show the value of $f(x)$ at the average of a and b is $\geq$ the average of $f(a)$ and $f(b)$. b) Prove the arithmetic-geometric mean inequality: $\sqrt{ab} \leq (a + b)/2$.

Proof. a), by definition of geometrically concave $\frac{f(x)-f(a)}{x-a} \geq \frac{f(b)-f(a)}{b-a}$ for any $a < x < b$ in I, take $x = \frac{a+b}{2}$, then

$$\frac{f(\frac{a+b}{2}) - f(a)}{\frac{b-a}{2}} \geq \frac{f(b) - f(a)}{b - a} \Rightarrow 2f(\frac{a + b}{2}) - 2f(a) \geq f(b) - f(a) \Rightarrow f(\frac{a + b}{2}) \geq \frac{f(a) + f(b)}{2}.$$

b). Let $f(x) = \ln(x)$, then $f'(x) = \frac{1}{x}$ is decreasing for $x > 0$, then $f(x)$ is concave, then it is geometrically concave. By the result of (a),

$$f(\frac{a + b}{2}) \geq \frac{f(a) + f(b)}{2} \Rightarrow \ln \frac{a + b}{2} \geq \frac{1}{2} \ln ab.$$

As $\ln x$ is increasing for $x > 0$, so $\frac{a+b}{2} \geq \sqrt{ab}$. □

Completeness: [0/-1 pts].