ASSIGNMENT 9

[3 pts] Q2 17.3-3: You want to estimate $\sin x$ to three decimal places over $|x| < 0.5$. How large should n be in order that the n-th order Taylor polynomial give you this accuracy over the given interval?

Proof. Since $\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$. By Cauchy's test for alternating series,

$$|s_n - \sin x| < \left| \frac{x^{2n+1}}{(2n+1)!} \right| < \left| \left(\frac{1}{2}\right)^{2n+1} \frac{1}{(2n+1)!} \right|, \qquad \text{for } |x| < 0.5,$$

Then $n = 2$ is enough to make the error less then 10^{-3}, which corresponding to order equals to 5, since the coefficient with order 4 is zero, so 3rd order Taylor polynomial is enough. □

[3 pts] Q4 18.3-2: Prove directly from the definition of integrability: on an interval [a,b], if $f(x)$ is differentiable and $f'(x)$ is bounded, then $f(x)$ is integrable.

Proof. Assume $|f'(x)| < M$, for any $\epsilon > 0$, let $\delta < \frac{\epsilon}{M(b-a)}$, then for any $\mathscr{P}$ such that $|\mathscr{P}| < \delta$,

$$|L_f(\mathscr{P}) - U_f(\mathscr{P})| \leq \sum_{i=1}^{n} (M_i - m_i)\, \Delta x_i \leq \sum_{i=1}^{n} |f'(\xi_i)| \Delta x_i \Delta x_i \leq M\delta(b-a) < \epsilon.$$

□

[4 pts] Q6 19.2-1: Evaluate $\int_0^1 e^x\, dx$ directly, by using (8) applied to the upper sums taken over the standard n-partition.

Proof. Let partition $\mathscr{P}_n = \{\frac{i}{n}\}_{i=0}^{n}$.

$$\lim_{n\to\infty} U(\mathscr{P}_n) = \lim_{n\to\infty} \sum_{i=1}^{n} e^{\frac{i}{n}} \frac{1}{n} = \lim_{n\to\infty} e^{1/n} \frac{1-e}{1-e^{1/n}} \frac{1}{n} = e - 1.$$

□

Challenge: Let $f(x) = e^{-1/x^2}$. Show that $f^{(n)}(0) = 0$ for all n.

Proof. Statement: for any n, the n-th order derivative of $f(x)$ has form as $f^{(n)}(x) = e^{-1/x^2} \sum_{i=2+n}^{3n} a_i^{(n)} \frac{1}{x^i}$ for some constant $a_i^{(n)}$. Prove it by induction, for $n = 1$, $f'(x) = \frac{2}{x^3} e^{-1/x^2}$ holds; Assume it also holds for $n = k$, which is $f^{(k)}(x) = e^{-1/x^2} \sum_{i=2+k}^{3k} a_i^{(k)} \frac{1}{x^i}$. then for $n = k+1$

$$f^{(k+1)}(x) = e^{-1/x^2} \frac{2}{x^3} \sum_{i=2+k}^{3k} a_i^{(k)} \frac{1}{x^i} + e^{-1/x^2} \sum_{i=2+k}^{3k} -i a_i^{(k)} \frac{1}{x^{i+1}} = e^{-1/x^2} \sum_{i=2+k+1}^{3k+3} a_i^{(k+1)} \frac{1}{x^i},$$

which prove the statement.

Since for any n, $f^{(n)}(x)$ has finite terms, and by L'Hospital rule, when $x \to 0$, all its terms have limitation to 0, then complete the proof. □

Completeness: [0/-1 pts].