MATH 490, WORKSHEET #6
WEDNESDAY, MARCH 20

Problem 1, IMO 1977. For a finite sequence of real numbers, every sum of seven successive terms is negative, while every sum of eleven successive terms is positive. What is the most terms such a sequence can have?

Problem 2, Putnam 1992 A deck of $2n$ cards numbered from 1 to $2n$ is shuffled and dealt to two players, Alex and Bernadette. Starting with Alex the players take turns discarding a card, face up. The game ends when the sum of the discards is first becomes divisible by $2n+1$, and the last player to discard wins. What is the probability that Alex wins if neither player makes a mistake?

Problem 3, Putnam 2001. There is a set consisting of n biased coins. The coin m has probability $\frac{1}{2m+1}$ of landing heads. What is the probability that if each coin is tossed once, you get an odd number of heads? Tosses are assumed to be independent.

Problem 4, ICMC 2007. (a) Let p be a prime. If an integer a is selected at random, what is the probability that a is divisible by p? Next suppose two integers are selected at random. What is the probability that both are divisible by p? Finally, suppose two integers are selected at random. Show that the probability that they are are relatively prime is $\prod_{p,\ \mathrm{prime}} \left(1 - \frac{1}{p^2}\right)$.

Problem 5, ICMC 2015. Show that

$$\sum_{k=0}^{n} \sum_{j=0}^{n-k} \binom{n}{k} \binom{n-k}{j} 2^{j+k} = 5^n.$$

Hint: Interpret this as the number of ways to pick two disjoint committees A and B out of n total people, and choosing a third, possibly empty, committee C from A and B.

Problem 6, ICMC 1996. Two couples agree to have n children each and plan to marry them off so that sons of one couple only marry daughters of the other. Assuming that sons and daughters are born with equal probability, find the probability that such a scheme is successful. Evaluate the asymptotic probability as $n \to \infty$. Hint: Stirling's formula.

Problem 7, Banach's matchbox problem. A man buys two boxes of matches, and places them in his pocket. Whenever he wants to light a match, he selects one matchbox at random. At some point the man finds one of the boxes is empty. What is the probability that there are k matches left in the other box? How does the help you evaluate

$$\sum_{k=1}^{n} 2^k \binom{2n-k}{n}?$$

Problem 8, Putnam 1989. A player plays the following game. At each turn a fair coin is tossed. Depending on the results of the tosses to date, (1) the game ends and the player wins, (2) the game ends and the player loses, or (3) the coin is tossed again. Given an irrational p in the interval $(0, 1)$, can we find a rule such that (A) the player wins with probability p, and (B) the game ends after a finite number of tosses with probability 1?