MATH 490, Worksheet #11, Wednesday, April 8, 2020

Problem 1. The Ulam numbers are defined as follows. $U_1 = 1$, $U_2 = 2$. If $U_1, \ldots, U_m$ are defined, then U_{m+1} is the next number greater than U_m which can be expressed as a unique sum of two distinct Ulam numbers. So $U_3 = 3 = 2 + 1$, $U_4 = 4 = 3 + 1$, $U_5 = 6 = 4 + 2$, etc. Show there are infinitely many Ulam numbers.

Problem 2. Narayana's cows is the sequence defined by $N_1 = 1, N_2 = 1, N_3 = 1$ and $N_k = N_{k-1} + N_{k-3}$. Show that $U_n \leqslant N_{n+2}$.

Problem 3, Engel. A sequence is given by $x_1 = 2, x_2 = 7$ and $x_n = 7x_{n-1} - 12x_{n-2}$. Find a closed form expression (formula) for x_n.

Problem 4, Engel. Is there an evenly spaced subsequence of $1/2, 1/4, 1/8, \ldots$ whose sum is $1/5$? How about $1/7$? (For example, every fifth term starting from the second term is evenly spaced.)

Problem 5. Consider the sequence a_n of numbers whose prime factors are only 2,3, and 5, that is $2, 3, 4, 5, 6, 8, 9, 10, 12, 15, \ldots$. Evaluate $\sum_{n=1}^{\infty} \frac{1}{a_n}$.

Problem 6, Engel. Find the sequence whose nth term is the number of permutations σ of $\{1, \ldots, n\}$ so that $|\sigma(k) - k| \leqslant 1$ for all $k = 1, \ldots, n$.

Problem 7. ICMC 2018. A sequence is defined by $x_1 = 1/2$, $x_{k+1} = x_k^2 + x_k$. Find the largest integer less than $\sum_{k=1}^{100} \frac{1}{x_k + 1}$.

Problem 8, Putnam 1990. Show that the sequence $a_1 = 2, a_2 = 3, a_3 = 6$ with relation $a_n = (n+4)a_{n-1} - 4na_{n-2} + (4n-8)a_{n-3}$ is the sum of two well-known sequences.

Narayana = Narayana Pandita. `https://en.wikipedia.org/wiki/Narayana_Pandita`

Ulam = Stanislaw Ulam. `https://en.wikipedia.org/wiki/Stanislaw_Ulam`

Engel = A. Engel, "Problem Solving Strategies," Springer, 1997.,

ICMC = Indiana Collegiate Mathematics Contest. `http://sections.maa.org/indiana/ICMC.php`