MATH 490, Worksheet #14, Wednesday, April 29, 2020

Problem 1. Show that $\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n} = 2^n$, $\binom{n}{1} + 2\binom{n}{2} + \cdots + n\binom{n}{n} = n2^{n-1}$

Problem 2. Show that $\binom{n}{0}^2 + \binom{n}{1}^2 + \cdots + \binom{n}{n}^2 = \binom{2n}{n}$. Show in general the Vandermonde identity that $\sum_{k=0}^{l} \binom{n}{k}\binom{m}{l-k} = \binom{n+m}{l}$.

Problem 3. Let $F_0 = 1, F_1 = 1, \ldots$ be the Fibonacci numbers. Show that $\sum_n F_n x^n$ is a rational function, i.e., a quotient of polynomials, and compute it.

Problem 4. Compute $\sum_{n=0}^{\infty} \frac{F_n}{2^n}$.

Problem 5. Find $\sum_{n=1}^{\infty} \frac{n}{2^n}$ and $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$.

Problem 6. Let C_n be the number of ways to cut a convex polygon with $n+2$ sides into n triangles. Cuts are not allowed to cross each other internally. Show that $C_{n+1} = \sum_{k=0}^{n} C_k C_{n-k}$. (We set $C_0 = 1$.)

Problem 7. Show that C_n is the number of legal expressions using n pairs of parentheses $($, $)$. C_n is known as the nth Catalan number.

Problem 8. Find the function $C(x) = \sum_{n=0}^{\infty} C_n x^n$.

Problem 9. Arrange $2n+1$ points on a circle. Assign the value 1 to $n+1$ points and the value -1 to the remaining n points. Show there is exactly one starting position so that as you add the number you encounter moving clockwise the sum stays positive.

Problem 10. Prove that $C_n = \frac{1}{n+1}\binom{2n}{n}$.