MATH 490, Worksheet #5, Wednesday, February 12, 2020

Problem 1, Engel. Six points are spaced evenly on a circle. These points are labeled $1, 0, 1, 0, 0, 0$ clockwise. You may increase or decrease any two neighboring numbers each by 1. Is there a way to equalize all the numbers?

Problem 2, Zeitz. A room is initially empty. During each minute either one person enters or two people leave. After 3^{2020} minutes can there be exactly $3^{212} + 2$ people in the room?

Problem 3, Zeitz. Let $P_1, \dots, P_{2019}$ be distinct points in the plane. Consider the line segments $P_1P_2, P_2P_3, \dots, P_{2018}P_{2019}, P_{2019}P_1$. Can a line be drawn in the plane that intersects the interior of each line segment?

Problem 4, Engel. For $a < b$ define sequences by $x_0 = a$, $y_0 = b$,

$$x_{n+1} = \frac{x_n + y_n}{2}, \; y_{n+1} = \frac{2x_n y_n}{x_n + y_n}.$$

Find $\lim_{n\to\infty} x_n$.

Problem 5, ICMC 1967. Let n be odd. If (a_{ij}) is a symmetric $n \times n$ matrix ($a_{ij} = a_{ji}$ for all i, j) such that each row is a permutation of $(1, \dots, n)$, then so is the main diagonal.

Problem 6, Engel. A rectangular floor is covered by 2×2 and 1×4 tiles. If one tile is smashed and replaced with one of the other kind, show that the tiles cannot be rearranged to cover the floor.

Problem 7, Engel. A $(2n + 1) \times (2n + 1)$ board has one square cut from a corner. For which values of n can the resulting shape be tiled with 2×1 tiles half of which are horizontal?

Problem 8, de Bruijn. A rectangle is tiled by smaller rectangles so that each of the smaller rectangles has a side of integral length. Show that the large rectangle must also have a side of integral length.

de Bruijn = `https://en.wikipedia.org/wiki/Nicolaas_Govert_de_Bruijn`

Engel = A. Engel, "Problem Solving Strategies," Springer, 1997.

ICMC = Indiana Collegiate Mathematics Contest. `http://sections.maa.org/indiana/ICMC.php`

Zeitz = P. Zeitz, "The Art and Craft of Problem Solving" 2 ed. Wiley, 2007.