MATH 490, Worksheet #8, Wednesday, March 4, 2020

Problem 1, Engel. n circles are drawn in the plane, dividing it into regions. Show that the regions may be colored black and gold so that no two regions which share a boundary line are given the same color.

Problem 2. A bracelet is made by stringing together $2n$ beads in some order, n red beads and n blue beads. Show that starting from some bead and moving counterclockwise that the number of red beads encountered is always at least the number of blue beads encountered.

Problem 3, Engel. There are n identical cars on a circular track. Among all of them, there is just enough gas for one car to complete a lap. Show that there is a car which can complete a lap by collecting gas from the other cars on its way around.

Problem 4. Tower of Hanoi. n disks of different radii are placed in a pile with largest on bottom. You may move the disk between 3 piles as long as you do not place a disk on top of one with smaller radius. Find the minimal number of moves it takes to move all of the disks from one pile to another.

Problem 5, Engel. How many subsets of $\{1, \ldots, n\}$ contain no two consecutive numbers?

Problem 6, Putnam 2015. Let $a_0 = 1$, $a_1 = 2$, and $a_n = 4a_{n-1} - a_{n-2}$. Find an odd prime factor of a_{2015}.

Problem 7, Putnam 1996. A set $S \subset \{1, \ldots, n\}$ is *selfish* if it contains its own cardinality as an element. Find the total number of minimal selfish subsets of $\{1, \ldots, n\}$.

Problem 8, ICMC 1998. Let (f_n) be the sequence of Fibonacci numbers. Show that (f_n, f_{n+1}) are exactly the positive integer points which belong to one of the hyperbolas $y^2 - xy - x^2 = \pm 1$.

Problem 9, Banach's Matchbox Problem. A mathematician carries two boxes of matches at all times, one in their left pocket and the other in their right pocket. They randomly pick a box and remove a match. At some point they open a matchbox and find that it is empty. (Mathematicians are very absent minded!) If both boxes started with n matches, what is the probability that the other mathbox contains k matches?

Banach = Stefan Banach. `https://en.wikipedia.org/wiki/Stefan_Banach`, Engel = A. Engel, "Problem Solving Strategies," Springer, 1997., ICMC = Indiana Collegiate Mathematics Contest. `http://sections.maa.org/indiana/ICMC.php`