MATH 490, Worksheet #9, Wednesday, March 25, 2020

Problem 1, Zeitz. For which integer n is $\frac{1}{n}$ closest to $\frac{1}{\sqrt{1{,}000{,}000}} - \frac{1}{\sqrt{999{,}999}}$

Problem 2. Minimize
$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z}$$
among all positive numbers x, y, z with $x + y + z = 1$.

Problem 3. Minimize
$$\sum_{i=1}^{n} \frac{1}{a_i}$$
among all positive $a_1, \ldots, a_n$ with $a_1 + \cdots + a_n = 1$.

Problem 4. Show that
$$\frac{1}{2\sqrt{n}} < \frac{1}{2} \cdot \frac{3}{4} \cdots \frac{2n-1}{2n} < \frac{1}{\sqrt{2n}}.$$

Problem 5. Zeitz Let $a_1, \ldots, a_n$ be positive numbers and $b_1, \ldots, b_n$ be a permutation of the first sequence. Minimize
$$\frac{a_1}{b_1} + \cdots + \frac{a_n}{b_n}.$$

Problem 6. Putnam 2003. Let $p(x) = ax^2 + bx + c$ and $q(x) = Ax^2 + Bx + C$ be quadratic polynomials (so $a, A \neq 0$). Suppose q has at least one real root. If $|p(x)| \leqslant |q(x)|$ for all x, show $|b^2 - 4ac| \leqslant |B^2 - 4AC|$. (This is actually still true if q has no real roots.)

Problem 7, Putnam 2003. For $a_1, \ldots, a_n, b_1, \ldots, b_n$ positive show that
$$(a_1 \cdots a_n)^{1/n} + (b_1 \cdots b_n)^{1/n} \leqslant ((a_1 + b_1) \cdots (a_n + b_n))^{1/n}.$$

Problem 8, Zeitz. Show that for x, y, z positive with $xyz = 1$ that
$$\frac{x^2}{y+z} + \frac{y^2}{x+z} + \frac{z^2}{x+y} \geqslant \frac{3}{2}.$$

Activity, Cauchy. Think about how to use that $\frac{a+b+c+d}{4} \geqslant (abcd)^{1/4}$ for any a, b, c, d positive to show that $\frac{x+y+z}{3} \geqslant (xyz)^{1/3}$ for all x, y, z positive. Use induction to show that the Arithmetic Mean - Geometric mean inequality is valid when taking the mean of 2^n and try to extend this reasoing to prove the general case.

Cauchy = `https://en.wikipedia.org/wiki/Augustin-Louis_Cauchy`

Zeitz = P. Zeitz, "The Art and Craft of Problem Solving" 2 ed. Wiley, 2007.