Introduction to ***dfield*** , ***pplane*** and `fplot`

It might be reasonable to expect that if two solutions of a differential equation which start close together, then they should stay relatively close together. We can investigate the behavior of solutions using the routine ***dfield*** .

(1) Use ***dfield***7 and the "Keyboard Input" feature to plot the solution to the differential equation $\boldsymbol{y' = y^2 - 2t}$ with $\boldsymbol{y(0) = 0.9185}$. Now using the "Zoom In" feature, approximate $y(3.2)$ to within 2 decimal places.

(2) Erase all solutions. Now use ***dfield***7 and the "Keyboard Input" feature to plot the solution to the differential equation $\boldsymbol{y' = y^2 - 2t}$ with $\boldsymbol{y(0) = 0.9184}$ (instead of $y(0) = 0.9185$). Use the "Zoom In" feature to approximate $y(3.2)$ to within 2 decimal places. Is this approximation close to the approximation in (1) ?

(3) Erase all solutions and using ***dfield***7 plot the two solutions to:

$$\begin{cases} y' = y^2 - 2t \\ \\ y(0) = 0.9185 \end{cases} \quad \text{and} \quad \begin{cases} y' = y^2 - 2t \\ \\ y(0) = 0.9184 \end{cases}$$

on a single graph. This shows the solutions start close. Do they remain close as $t \to \infty$?

<u>Remark</u>: This example illustrates *Chaos Theory*: a small change in the initial condition can lead to very different outcomes. This specific example shows that if you are a mere 0.0001 off in your measurement of $y(0)$, you could arrive at drastically different solutions. This is one reason why weather prediction for more than a few days into the future is so difficult.

(4) The routine ***pplane***7 is similar to ***dfield***7, except it is used to sketch solutions to *systems* of differential equations of the form

$$\frac{dx}{dt} = F(t, x, y)$$

$$\frac{dy}{dt} = G(t, x, y)$$

Use ***pplane*** and the "Keyboard Input" feature to plot solutions to the system

$$x' = 2y + 1$$

$$y' = -\frac{3}{1+x}$$

with $x = 0$ and various values of $y = y_0$. Estimate the *smallest* value of y_0 so that when $x = 0$ and $y = y_0$, the solution satisfies $y \to \infty$ (i.e., the solution y is unbounded).

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(5) The solution to the initial value problem

$$y' = \frac{(1-2x)}{e^{2x}} + 3x^2 - \pi \sin \pi x\,, \quad y(0) = 2$$

is given by

$$y = xe^{-2x} + x^3 + \cos \pi x + 1\,.$$

Use **fplot** to determine the value(s) of x (correct within 2 decimal places) so that

$$y = \textbf{seed}\,.$$

(For example, the command "`>> fplot('44*T+379',[0,20,0,7000],'g')`" will plot the function $P(t) = 44t + 379$ in green.) Use ***dfield*** to check you answer.

(6) **Challenge problem**: The solution $y = y(x)$ to the initial value problem

$$\frac{dy}{dx} = \frac{2y - 3x^2}{2(y-x)}\,, \quad y(1) = 0$$

is given (implicitly) by

$$x^3 + y^2 - 2xy = 1\,.$$

Approximate $y(-1)$ correct to 2 decimal places using:

(a) ***dfield***

(b) `fplot`

Hint: Complete the square with respect to y.

Even though ***pplane*** is used for systems of differential equations (and for higher order differential equations as you'll see later), one can use ***pplane*** to find solutions to a single differential equation.

- Use ***dfield*** , with window $-1 \le t \le 1,\ -1 \le x \le 1$, to plot the solutions of the differential equation $x' = 2xe^t - t^2$ with these initial conditions:
  $(t, x) = (0, -1),\ (0, -0.6),\ (0, 0),$ and $(0, \frac{1}{2})$.

  Print one graph with all 4 solutions on it.

- Now use ***pplane*** to plot solutions to the system

$$\begin{aligned} y' &= 1 \\ x' &= 2xe^y - y^2 \end{aligned}$$

  (make sure $y' = 1$ is the first equation, else the axes will be reversed on your graph) using the window $-1 \le y \le 1,\ -1 \le x \le 1$ and initial conditions:
  $(y, x) = (0, -1),\ (0, -0.6),\ (0, 0),$ and $(0, \frac{1}{2})$.

  Do these solutions match those above ? Can you explain why ?