

NONEXISTENCE OF ARITHMETIC FAKE COMPACT HERMITIAN
SYMMETRIC SPACES OF TYPES B_n , C_n , D_n , E_6 AND E_7

GOPAL PRASAD AND SAI-KEE YEUNG

1. Introduction

1.1. Let $\overline{\mathcal{G}}$ be a noncompact connected real semi-simple Lie group with trivial center and with no nontrivial compact connected normal subgroups, and \mathfrak{g} be its Lie algebra. The group $\text{Aut}(\overline{\mathcal{G}})$ ($=\text{Aut}(\mathfrak{g})$) of automorphisms of $\overline{\mathcal{G}}$ is a Lie group with finitely many connected components, and $\overline{\mathcal{G}}$ is its identity component. We will denote the identity component of $\text{Aut}(\overline{\mathcal{G}})$ in the *Zariski-topology* by $\text{Int}(\overline{\mathcal{G}})$. Let X be the symmetric space of $\overline{\mathcal{G}}$ (X is the space of maximal compact subgroups of $\overline{\mathcal{G}}$), and X_u be the compact dual of X . There is a natural identification of the group of isometries of X with $\text{Aut}(\overline{\mathcal{G}})$. We assume in this paper that X (and hence X_u) is hermitian. Then every holomorphic automorphism of X is an isometry. The group $\text{Hol}(X)$ of holomorphic automorphisms of X is a subgroup of finite index of the group $\text{Aut}(\overline{\mathcal{G}})$ of isometries, and it is known (see [Ta], the remark in §5) that $\text{Hol}(X) \cap \text{Int}(\overline{\mathcal{G}}) = \overline{\mathcal{G}}$.

1.2. We will say that the quotient X/Π of X by a *torsion-free* cocompact discrete subgroup Π of $\overline{\mathcal{G}}$ is a *fake compact hermitian symmetric space*, or a *fake X_u* , if its Betti numbers are same as that of X_u ; X/Π is an *arithmetic fake compact hermitian symmetric space*, or an *arithmetic fake X_u* , if, further, Π is irreducible (i.e., no subgroup of Π of finite index is a direct product of two infinite normal subgroups) and it is an arithmetic subgroup of $\overline{\mathcal{G}}$. Any such space can be endowed with the structure of a smooth complex projective variety. Several such spaces have been constructed in our two earlier papers [PY1] and [PY2]. In [PY1] we have given a classification of “fake projective planes”, the first of which was constructed by David Mumford in [Mu] using p -adic uniformization. In [PY2] we have constructed four arithmetic fake $\mathbf{P}_{\mathbb{C}}^4$, four arithmetic fake Grassmannians $\mathbf{Gr}_{2,5}$, and five (irreducible) arithmetic fake $\mathbf{P}_{\mathbb{C}}^2 \times \mathbf{P}_{\mathbb{C}}^2$. All these are Shimura varieties.

We note that if $\overline{\mathcal{G}}$ contains an irreducible arithmetic subgroup, then the simple factors of its complexification are isomorphic to each other, see [Ma], Corollary 4.5 in Ch. IX. Also, if the real rank of $\overline{\mathcal{G}}$ is at least 2, then by Margulis’ arithmeticity theorem ([Ma], Ch. IX), any irreducible discrete cocompact subgroup of $\overline{\mathcal{G}}$ (in fact, any irreducible lattice) is arithmetic.

If Π is a torsion-free cocompact discrete subgroup of $\overline{\mathcal{G}}$, then there is a natural embedding of $H^*(X_u, \mathbb{C})$ in $H^*(X/\Pi, \mathbb{C})$, see [B], 3.1 and 10.2, and hence X/Π is a fake X_u if and only if the natural homomorphism $H^*(X_u, \mathbb{C}) \rightarrow H^*(X/\Pi, \mathbb{C})$ is an isomorphism.

1.3. Let $\overline{\mathcal{G}}$, X and X_u be as above, and let Π be a torsion-free cocompact discrete subgroup of $\overline{\mathcal{G}}$. Let $Z = X/\Pi$. If Z is a fake X_u , then the Euler-Poincaré characteristic $\chi(Z)$ of $Z = X/\Pi$, and so the Euler-Poincaré characteristic $\chi(\Pi)$ of Π equals $\chi(X_u)$. As X has been assumed to be hermitian, the Euler-Poincaré characteristic of X_u is positive. On the other hand, it follows from Hirzebruch proportionality principle, see [S], Proposition 23, that the Euler-Poincaré characteristic of X/Π is positive if and only if the complex dimension of X is even. Using the results of [BP], we can easily conclude that there are only finitely many irreducible arithmetic fake compact hermitian symmetric spaces. It is of interest to determine them all.

1.4. Hermitian symmetric spaces have been classified by Élie Cartan; see [H], Ch. IX. We recall that the irreducible hermitian symmetric spaces are the symmetric spaces of Lie groups $SU(n+1-m, m)$, $SO(2, 2n-1)$, $Sp(2n)$, $SO(2, 2n-2)$, $SO^*(2n)$, an absolutely simple real Lie group of type E_6 with Tits index ${}^2E_{6,2}^{16'}$, and an absolutely simple real Lie group of type E_7 with Tits index $E_{7,3}^{28}$ respectively (for Tits indices see Table II in [Ti1]). The complex dimensions of these spaces are $(n+1-m)m$, $2n-1$, $n(n+1)/2$, $2n-2$, $n(n-1)/2$, 16 and 27 respectively. The Lie groups listed above are of type A_n , B_n , C_n , D_n , D_n , E_6 and E_7 respectively. We will say that an irreducible symmetric space is one of these types if it is the symmetric space of a simple Lie group of that type, and say that a locally hermitian symmetric space is of one of these types if its simply connected cover is a product of irreducible hermitian symmetric spaces of that type.

The following is the main result of this paper.

Theorem. *There does not exist an irreducible arithmetic fake compact hermitian symmetric space of type B_n , C_n , D_n , E_6 or E_7 .*

In the following subsection we will explain the strategy of the proof, and fix notation which will be used throughout the paper.

1.5. Let $\overline{\mathcal{G}}$, X , X_u be as in 1.1; X will be assumed to be a hermitian symmetric space of one of the following types B_n , C_n , E_6 and E_7 . Assume, if possible, that $\overline{\mathcal{G}}$ contains a cocompact irreducible arithmetic subgroup Π whose orbifold Euler-Poincaré characteristic $\chi(\Pi)$ equals $\chi(X_u)$. Then there exist a totally real number field k , a connected adjoint absolutely simple algebraic k -group \overline{G} of same type as X , \overline{G} of k -rank 0 (by Godement criterion since Π is cocompact), real places v_1, \dots, v_r of k such that $\overline{G}(k_v)$ is compact for every real place v different from v_1, \dots, v_r , $\overline{\mathcal{G}}$ is isomorphic to $\prod_{j=1}^r \overline{G}(k_{v_j})^o$ (and will be identified with it), and Π is commensurable with an arithmetic subgroup of $\overline{G}(k)$. Let $\pi : G \rightarrow \overline{G}$ be the simply connected covering of \overline{G} defined over k . The kernel of the isogeny π is the center C of the simply connected k -group G . If G is of type E_6 , then it is an “outer k -form” of a split group (i.e., it is of type 2E_6) since X is hermitian. In this case let ℓ be the quadratic extension of k over which G is an “inner k -form”. Then, as the unique \mathbb{R} -anisotropic form of type E_6 is also an outer form, ℓ is totally complex.

Description of C : For a positive integer s , let μ_s be the kernel of the endomorphism $x \mapsto x^s$ of GL_1 . Then if G is of type 2E_6 , its center C is k -isomorphic to

the kernel of the norm map $N_{\ell/k}$ from the algebraic group $R_{\ell/k}(\mu_3)$, obtained from μ_3 by Weil's restriction of scalars, to μ_3 . If G is of type B_n , C_n or E_7 , then C is isomorphic to μ_2 .

It is known, and easy to see using the above description of C , that for any real place v of k , the order of the kernel of the induced homomorphism $G(k_v) \rightarrow \overline{G}(k_v)$ is 2 if G is not of type 2E_6 , and of order 3 if it is of type 2E_6 . Moreover, as $G(k_v)$ is connected, $\pi(G(k_v)) = \overline{G}(k_v)^o$. Let $\mathcal{G} = \prod_{j=1}^r G(k_{v_j})$, and let $\tilde{\Pi}$ be the inverse image of Π in \mathcal{G} . Then the kernel of the homomorphism $\pi : \mathcal{G} \rightarrow \overline{\mathcal{G}}$ is of order s^r , and hence the orbifold Euler-Poincaré characteristic $\chi(\tilde{\Pi})$ of $\tilde{\Pi}$ equals $\chi(\Pi)/s^r = \chi(X_u)/s^r$, where, here and in the sequel, $s = 2$ if G is not of type E_6 , and $s = 3$ if G is of type E_6 . Now let Γ be a maximal discrete subgroup of \mathcal{G} containing $\tilde{\Pi}$. Then the orbifold Euler-Poincaré characteristic $\chi(\Gamma)$ of Γ is a submultiple of $\chi(\tilde{\Pi}) = \chi(X_u)/s^r$. Using the volume formula of [P], some number theoretic estimates, the Bruhat-Tits theory, and the Hasse principle for semi-simple groups (Proposition 7.1 of [PR]), we will prove that \mathcal{G} does not contain such a subgroup.

2. Preliminaries

2.1. We will use the notations introduced in 1.5. Thus k will be a totally real number field, G an absolutely simple simply connected algebraic k -group (of one of the following four types: B_n , C_n , 2E_6 , and E_7), C its center, $\mathcal{G} = \prod_{j=1}^r G(k_{v_j})$. We will think of $G(k)$ as a subgroup of \mathcal{G} in terms of its diagonal embedding. Γ is a maximal arithmetic subgroup of \mathcal{G} (arithmetic with respect to the k -structure on G) whose orbifold Euler-Poincaré characteristic is a submultiple of $\chi(X_u)/s^r$. Then $\Lambda := \Gamma \cap G(k)$ is a ‘‘principal’’ arithmetic subgroup, i.e., for every nonarchimedean place v of k , the closure P_v of Λ in $G(k_v)$ is a parahoric subgroup, $\Lambda = G(k) \cap \prod_{v \in V_f} P_v$, and Γ is the normalizer of Λ in \mathcal{G} , see Proposition 1.4(iv) of [BP]. Let the ‘‘type’’ Θ_v of P_v be as in [BP], 2.2, and Ξ_{Θ_v} be as in 2.8 there. If either P_v is hyperspecial, or G is of type 2E_6 and it does not split over k_v , then Ξ_{Θ_v} is trivial. The order of Ξ_{Θ_v} is always a divisor of s .

In terms of the normalized Haar-measure μ on $\mathcal{G} = \prod_{j=1}^r G(k_{v_j})$ used in [P] and [BP], and to be used in this paper, $\chi(\Gamma) = \chi(X_u)\mu(\mathcal{G}/\Gamma)$ (see [BP], 4.2). Thus the condition that $\chi(\Gamma)$ is a submultiple of $\chi(X_u)/s^r$ is equivalent to the condition that $\mu(\mathcal{G}/\Gamma)$ is a submultiple of $1/s^r$. We will prove that there does not exist such a Γ .

A comprehensive survey of the basic notions and the main results of the Bruhat-Tits theory of reductive groups over nonarchimedean local fields is given in [Ti2].

2.2. All unexplained notations are as in [BP] and [P]. Thus for a number field K , D_K will denote the absolute value of its discriminant, h_K its class number, i.e., the order of its class group $Cl(K)$. We will denote by $h_{K,s}$ the order of the subgroup of $Cl(K)$ consisting of the elements of order dividing s , where, as in 1.5, $s = 2$ if G is not of type E_6 , and $s = 3$ if G is of type E_6 . Then $h_{K,s} \leq h_K$. We will denote by U_K the multiplicative-group of units of K , and by K_s the subgroup of K^\times consisting of the elements x such that for every normalized valuation v of K , $v(x) \in s\mathbb{Z}$.

V_f (resp. V_∞) will denote the set of nonarchimedean (resp. archimedean) places of k . As k admits at least r distinct real places, see 1.5, $d := [k : \mathbb{Q}] \geq r$. For $v \in V_f$, q_v will denote the cardinality of the residue field \mathfrak{f}_v of k_v .

2.3. For a parahoric subgroup P_v of $G(k_v)$, we define $e(P_v)$ and $e'(P_v)$ by the following formulae (cf. Theorem 3.7 of [P]):

$$(1) \quad e(P_v) = \frac{q_v^{(\dim \bar{M}_v + \dim \bar{\mathcal{M}}_v)/2}}{\#\bar{M}_v(\mathfrak{f}_v)}.$$

$$(2) \quad e'(P_v) = e(P_v) \cdot \frac{\#\bar{\mathcal{M}}_v(\mathfrak{f}_v)}{\dim \bar{\mathcal{M}}_v} = q_v^{(\dim \bar{M}_v - \dim \bar{\mathcal{M}}_v)/2} \cdot \frac{\#\bar{\mathcal{M}}_v(\mathfrak{f}_v)}{\#\bar{M}_v(\mathfrak{f}_v)}.$$

2.4. Let m_1, \dots, m_n ($m_1 < \dots < m_n$), where n is the absolute rank of G , be the exponents of the Weyl group of G . For type B_n and C_n , $m_j = 2j - 1$; for type E_6 , the exponents are 1, 4, 5, 7, 8 and 11; and for type E_7 , the exponents are 1, 5, 7, 9, 11, 13 and 17. Then

- if either G is *not* of type 2E_6 , or v splits in ℓ ,

$$e'(P_v) = e(P_v) \prod_{j=1}^n \left(1 - \frac{1}{q_v^{m_j+1}}\right);$$

- if G is of type 2E_6 and v does not split in ℓ ,

$$e'(P_v) = e(P_v) \left(1 - \frac{1}{q_v^2}\right) \left(1 + \frac{1}{q_v^5}\right) \left(1 - \frac{1}{q_v^6}\right) \left(1 - \frac{1}{q_v^8}\right) \left(1 + \frac{1}{q_v^9}\right) \left(1 - \frac{1}{q_v^{12}}\right),$$

or

$$e'(P_v) = e(P_v) \left(1 - \frac{1}{q_v^2}\right) \left(1 - \frac{1}{q_v^6}\right) \left(1 - \frac{1}{q_v^8}\right) \left(1 - \frac{1}{q_v^{12}}\right)$$

according as v *does not* or *does* ramify in ℓ .

2.5. It is obvious that $e'(P_v) < e(P_v)$. It is not difficult to check using (2) that *for all $v \in V_f$, and an arbitrary parahoric subgroup P_v of $G(k_v)$, $e'(P_v)$ is an integer.*

2.6. Now we will use the volume formula of [P] to write down the precise value of $\mu(\mathcal{G}/\Lambda)$. As the Tamagawa number $\tau_k(G)$ of G equals 1, Theorem 3.7 of [P] (recalled in 3.7 of [BP]), for $S = V_\infty$, provides us the following if G is *not* of type 2E_6 ,

$$(3) \quad \mu(\mathcal{G}/\Lambda) = D_k^{\frac{1}{2} \dim G} \left(\prod_{j=1}^n \frac{m_j!}{(2\pi)^{m_j+1}} \right)^d \mathcal{E},$$

and if G is of type 2E_6 ,

$$(4) \quad \mu(\mathcal{G}/\Lambda) = (D_k D_\ell)^{13} \left(\frac{4!5!7!8!11!}{(2\pi)^{42}} \right)^d \mathcal{E},$$

where n is the absolute rank of G , and $\mathcal{E} = \prod_{v \in V_f} e(P_v)$, with $e(P_v)$ as in 2.1.

2.7. Let ζ_k be the Dedekind zeta-function of k , and $L_{\ell|k}$ be the Hecke L -function associated to the quadratic Dirichlet character of ℓ/k . Then

$$\zeta_k(a) = \prod_{v \in V_f} \left(1 - \frac{1}{q_v^a}\right)^{-1};$$

$$L_{\ell|k}(a) = \prod' \left(1 - \frac{1}{q_v^a}\right)^{-1} \prod'' \left(1 + \frac{1}{q_v^a}\right)^{-1},$$

where \prod' is the product over the nonarchimedean places v of k which split in ℓ , and \prod'' is the product over all the other nonarchimedean places v which do not ramify in ℓ . Hence the Euler product \mathcal{E} appearing in (3) can be rewritten as

$$(5) \quad \mathcal{E} = \prod_{v \in V_f} e'(P_v) \prod_{j=1}^n \zeta_k(m_j + 1),$$

and the one appearing in (4) can be rewritten as

$$(6) \quad \mathcal{E} = \prod_{v \in V_f} e'(P_v) \cdot \zeta_k(2) L_{\ell|k}(5) \zeta_k(6) \zeta_k(8) L_{\ell|k}(9) \zeta_k(12).$$

2.8. If G is *not* of type 2E_6 , let \mathcal{J} be the set of all $v \in V_f$ such that P_v is not a hyperspecial parahoric subgroup of $G(k_v)$, and \mathcal{J}' be the empty set. If G is of type 2E_6 , let \mathcal{J} be the set of all $v \in V_f$ which splits in ℓ and P_v is not a hyperspecial parahoric subgroup, and let \mathcal{J}' be the set of $v \in V_f$ which does not split in ℓ , and either P_v is not a hyperspecial parahoric subgroup of $G(k_v)$ but v is unramified over ℓ , or v is ramified in ℓ and P_v is not a special parahoric subgroup. Then for all nonarchimedean $v \notin \mathcal{J}$, Ξ_{Θ_v} is trivial; if $v \notin \mathcal{J} \cup \mathcal{J}'$, $e'(P_v) = 1$, and $e'(P_v) > s$ if $v \in \mathcal{J}$. Therefore, $1 \leq e'(P_v) < e(P_v)$, and $\mathcal{E} = \prod_{v \in V_f} e(P_v) > s^{\#\mathcal{J}}$. Hence, if G is not of type 2E_6 ,

$$(7) \quad \mu(\mathcal{G}/\Lambda) > D_k^{\frac{1}{2} \dim G} \left(\prod_{j=1}^n \frac{m_j!}{(2\pi)^{m_j+1}} \right)^d 2^{\#\mathcal{J}},$$

and if G is of type 2E_6 ,

$$(8) \quad \mu(\mathcal{G}/\Lambda) > (D_k D_\ell)^{13} \left(\frac{4!5!7!8!11!}{(2\pi)^{42}} \right)^d 3^{\#\mathcal{J}}.$$

2.9. If G is not of type 2E_6 , let

$$(9) \quad \mathcal{R} = D_k^{\frac{1}{2} \dim G} \left(\prod_{j=1}^n \frac{m_j!}{(2\pi)^{m_j+1}} \right)^d \prod_{j=1}^n \zeta_k(m_j + 1),$$

and if G is of type 2E_6 , let

$$(10) \quad \mathcal{R} = (D_k D_\ell)^{13} \left(\frac{4!5!7!8!11!}{(2\pi)^{42}} \right)^d \zeta_k(2) L_{\ell|k}(5) \zeta_k(6) \zeta_k(8) L_{\ell|k}(9) \zeta_k(12).$$

Then

$$(11) \quad \mu(\mathcal{G}/\Lambda) = \mathcal{R} \prod_{v \in \mathcal{J} \cup \mathcal{J}'} e'(P_v).$$

As $e'(P_v)$ is an integer for all v (see 2.5), we conclude that $\mu(\mathcal{G}/\Lambda)$ is an integral multiple of \mathcal{R} .

Using the functional equations

$$\zeta_k(2a) = D_k^{\frac{1}{2}-2a} \left(\frac{(-1)^a 2^{2a-1} \pi^{2a}}{(2a-1)!} \right)^d \zeta_k(1-2a),$$

and

$$L_{\ell|k}(2a+1) = \left(\frac{D_k}{D_\ell} \right)^{2a+\frac{1}{2}} \left(\frac{(-1)^a 2^{2a} \pi^{2a+1}}{(2a)!} \right)^d L_{\ell|k}(-2a),$$

for every positive integer a , and the fact that $\dim G = n + 2 \sum m_j$, we find that if G is not of type 2E_6 ,

$$(12) \quad \mathcal{R} = 2^{-dn} \left| \prod_{j=1}^n \zeta_k(-m_j) \right|,$$

and if G is of type 2E_6 ,

$$(13) \quad \mathcal{R} = 2^{-6d} \zeta_k(-1) L_{\ell|k}(-4) \zeta_k(-5) \zeta_k(-7) L_{\ell|k}(-8) \zeta_k(-11).$$

2.10. As $\chi(\Lambda) = \chi(X_u) \mu(\mathcal{G}/\Lambda)$ ([BP], 4.2), we have the following

$$(14) \quad \chi(\Gamma) = \frac{\chi(\Lambda)}{[\Gamma : \Lambda]} = \frac{\chi(X_u) \mu(\mathcal{G}/\Lambda)}{[\Gamma : \Lambda]}.$$

Proposition 2.9 of [BP] applied to $G' = G$ and $\Gamma' = \Gamma$ implies that $[\Gamma : \Lambda]$ is a power of the prime number s . Now since $\mu(\mathcal{G}/\Lambda)$ is an integral multiple of \mathcal{R} , we conclude from (10) that if $\chi(\Gamma)$ is a submultiple of $\chi(X_u)$, then the numerator of the rational number \mathcal{R} is a power of s . We state this as the following proposition.

Proposition 1. *If the orbifold Euler-Poincaré characteristic of Γ is a submultiple of $\chi(X_u)$, then the numerator of the rational number \mathcal{R} is a power of s .*

2.11. In this paragraph we assume that G is not of type 2E_6 . Then $C \cong \mu_2$, and the Galois cohomology $H^1(k, C) \cong k^\times/k^{\times 2}$. The order of the first term of the short exact sequence of Proposition 2.9 of [BP], for $G' = G$ and $S = V_\infty$, is 2^{r-1} . From the proof of Proposition 0.12 of [BP], we easily conclude that $\#k_2/k^{\times 2} \leq h_{k,2} 2^d$. Now we can adapt the argument used to prove Proposition 5.1, and the argument in 5.5, of [BP], for $S = V_\infty$ and $G' = G$, to derive the following bound:

$$(15) \quad [\Gamma : \Lambda] \leq 2^{d+r-1+\#\mathcal{J}} h_{k,2}.$$

2.12. We shall assume now that G is of type 2E_6 . As the norm map $N_{\ell/k} : \mu_3(\ell) \rightarrow \mu_3(k)$ is onto, the Galois cohomology group $H^1(k, C)$ is isomorphic to the kernel of the homomorphism $\ell^\times/\ell^{\times 3} \rightarrow k^\times/k^{\times 3}$ induced by the norm map. We shall denote this kernel by $(\ell^\times/\ell^{\times 3})_\bullet$.

By Dirichlet's unit theorem, $U_k \cong \{\pm 1\} \times \mathbb{Z}^{d-1}$, and $U_\ell \cong \mu(\ell) \times \mathbb{Z}^{d-1}$, where $\mu(\ell)$ is the finite cyclic group of roots of unity in ℓ . Hence, $U_k/U_k^3 \cong (\mathbb{Z}/3\mathbb{Z})^{d-1}$, and $U_\ell/U_\ell^3 \cong \mu(\ell)_3 \times (\mathbb{Z}/3\mathbb{Z})^{d-1}$, where $\mu(\ell)_3$ is the group of cube-roots of unity in ℓ . Now we observe that $N_{\ell/k}(U_\ell) \supset N_{\ell/k}(U_k) = U_k^2$, which implies that the homomorphism $U_\ell/U_\ell^3 \rightarrow U_k/U_k^3$, induced by the norm map, is onto. Therefore, the order of the kernel $(U_\ell/U_\ell^3)_\bullet$ of this homomorphism equals $\#\mu(\ell)_3$.

The short exact sequence (4) in the proof of Proposition 0.12 of [BP] gives us the following exact sequence:

$$1 \rightarrow (U_\ell/U_\ell^3)_\bullet \rightarrow (\ell_3/\ell^{\times 3})_\bullet \rightarrow (\mathcal{P} \cap \mathcal{J}^3)/\mathcal{P}^3,$$

where $(\ell_3/\ell^{\times 3})_\bullet = (\ell_3/\ell^{\times 3}) \cap (\ell^\times/\ell^{\times 3})_\bullet$, \mathcal{P} is the group of all fractional principal ideals of ℓ , and \mathcal{J} the group of all fractional ideals (we use multiplicative notation for the group operation in both \mathcal{J} and \mathcal{P}). Since the order of the last group of the above exact sequence is $h_{\ell,3}$, see (5) in the proof of Proposition 0.12 of [BP], we conclude that

$$\#(\ell_3/\ell^{\times 3})_\bullet \leq \#\mu(\ell)_3 \cdot h_{\ell,3}.$$

Now we note that the order of the first term of the short exact sequence of Proposition 2.9 of [BP], for $G' = G$ and $S = V_\infty$, is $3^r/\#\mu(\ell)_3$.

Using the above observations, together with Proposition 2.9 and Lemma 5.4 of [BP], and a close look at the arguments in 5.3 and 5.5 of [BP] for $S = V_\infty$ and G as above, we can derive the following upper bound:

$$(16) \quad [\Gamma : \Lambda] \leq 3^{r+\#\mathcal{J}} h_{\ell,3}.$$

2.13. Since $\mu(\mathcal{G}/\Gamma) = \mu(\mathcal{G}/\Lambda)/[\Gamma : \Lambda]$ is a submultiple of $1/s^r$ (see 2.1), we conclude that $\mu(\mathcal{G}/\Lambda) \leq [\Gamma : \Lambda]/s^r$. From the bound for $[\Gamma : \Lambda]$ derived in 2.11 and 2.12 we obtain that if G is not of type 2E_6 , then

$$(17) \quad \mu(\mathcal{G}/\Lambda) \leq 2^{d-1+\#\mathcal{J}} h_{k,2},$$

and if G is of type 2E_6 ,

$$(18) \quad \mu(\mathcal{G}/\Lambda) \leq 3^{\#\mathcal{J}} h_{\ell,3}.$$

Now combining these with (7) and (8) respectively, we obtain

$$(19) \quad D_k^{\frac{1}{2}\dim G} < 2^{d-1} h_{k,2} \left(\prod_{j=1}^n \frac{(2\pi)^{m_j+1}}{m_j!} \right)^d,$$

if G is not of type 2E_6 , and

$$(20) \quad (D_k D_\ell)^{13} < h_{\ell,3} \left(\frac{(2\pi)^{42}}{4!5!7!8!11!} \right)^d$$

if G is of type 2E_6 .

3. Discriminant bounds

We will recall discriminant bounds required in later discussions. We define $M_r(d) = \min_K D_K^{1/d}$, where the minimum is taken over all totally real number fields K of degree d . Similarly, we define $M_c(d) = \min_K D_K^{1/d}$, by taking the minimum over all totally complex number fields K of degree d .

The precise values of $M_r(d), M_c(d)$ for low values of d are given in the following table (cf. [N]).

d	2	3	4	5	6	7	8
$M_r(d)^d$	5	49	725	14641	300125	20134393	282300416
$M_c(d)^d$	3		117		9747		1257728

The following proposition can be proved in the same way as Proposition 2 in [PY2] has been proved.

Proposition 2. Let k and ℓ be a totally real number field and a totally imaginary number field of degree d and $2d$ respectively.

$\forall d \geq$	2	3	4	5	6	7	8
$D_k^{1/d} >$	2.23	3.65	5.18	6.8	8.18	11.05	11.38
$D_\ell^{1/2d} >$	1.73		3.28		4.62		5.78

4. G of type B_n or C_n

4.1. In this section we assume that G is either of type B_n or C_n . Then its dimension is $n(2n+1)$. The j -th exponent $m_j = 2j-1$, and the complex dimension of the symmetric space X of $\mathcal{G} = \prod_{j=1}^r G(k_{v_j})$ is $r(2n-1)$ if G is of type B_n , and is $rn(n+1)/2$ if G is of type C_n . From (19) we obtain

$$(21) \quad D_k^{1/d} < f_1(n, d, h_{k,2}) := \left[\left\{ 2 \prod_{j=1}^n \frac{(2\pi)^{2j}}{(2j-1)!} \right\}^d \cdot \frac{h_{k,2}}{2} \right]^{2/dn(2n+1)}.$$

According to the Brauer-Siegel Theorem, for a totally real number field k , and all real $\delta > 0$,

$$h_k R_k \leq 2^{1-d} \delta (1+\delta) \Gamma((1+\delta)/2)^d (\pi^{-d} D_k)^{(1+\delta)/2} \zeta_k(1+\delta),$$

where R_k is the regulator of k . Now from (21) we get the following bound:

$$(22) \quad D_k^{1/d} < f_2(n, d, R_k, \delta) := \left[\left\{ \frac{\Gamma((1+\delta)/2) \zeta(1+\delta)}{\pi^{(1+\delta)/2}} \prod_{j=1}^n \frac{(2\pi)^{2j}}{(2j-1)!} \right\} \cdot \left\{ \frac{\delta(1+\delta)}{R_k} \right\}^{1/d} \right]^{2/(2n^2+n-1-\delta)},$$

since $\zeta_k(1+\delta) \leq \zeta(1+\delta)^d$, where $\zeta = \zeta_{\mathbb{Q}}$. Using the lower bound $R_k \geq 0.04 e^{0.46d}$, for a totally real number field k , due to R. Zimmert [Z], we get

$$(23) \quad D_k^{1/d} < f_3(n, d, \delta)$$

$$:= \left[\left\{ \frac{\Gamma((1+\delta)/2)\zeta(1+\delta)}{\pi^{(1+\delta)/2}e^{0.46}} \prod_{j=1}^n \frac{(2\pi)^{2j}}{(2j-1)!} \right\} \cdot \{25\delta(1+\delta)\}^{1/d} \right]^{2/(2n^2+n-1-\delta)}.$$

4.2. It is obvious that for fixed n and $\delta \in [0.04, 9]$, $f_3(n, d, \delta)$ decreases as d increases. Now we observe that for $n \geq 9$, $(2n-1)! > (2\pi)^{2n}$. From this it is easy to see that if for a given d, δ , and $n \geq 8$, $f_3(n, d, \delta) \geq 1$, then $f_3(n+1, d, \delta) < f_3(n, d, \delta)$, and if $f_3(n, d, \delta) < 1$, then $f_3(n+1, d, \delta) < 1$. In particular, if for given d , and $\delta \in [0.04, 9]$, $f_3(8, d, \delta) < c$, with $c \geq 1$, then $f_3(n, d', \delta) < c$ for all $n \geq 8$ and $d' \geq d$.

We obtain by a direct computation the following upper bound for the value of $f_3(n, 2, 3)$ for $6 \leq n \leq 14$.

$$\begin{array}{cccccccccc} n & 14 & 13 & 12 & 11 & 10 & 9 & 8 & 7 & 6 \\ f_3(n, 2, 3) < & 1 & 1.1 & 1.2 & 1.3 & 1.4 & 1.6 & 1.8 & 2.1 & 2.4 \end{array}$$

From the bounds provided by the above table and the properties of f_3 we conclude that $f_3(n, d, 3) < 2.1$ for all $n \geq 7$, and $d \geq 2$, and we conclude from Proposition 2 that unless $k = \mathbb{Q}$ (i.e., $d = 1$), $n \leq 6$.

We assert now that $n \leq 13$. To prove this, we can assume, in view of the result established in the preceding paragraph, that $k = \mathbb{Q}$. By a direct computation we see that $f_1(14, 1, 1) < 1$. Hence, $f_1(n, 1, 1) < 1$ for all $n \geq 14$. As $D_{\mathbb{Q}} = 1$, from bound (21) we conclude that $n \leq 13$.

We will now assume that $d \geq 2$ and consider each of the possible cases $2 \leq n \leq 6$ separately.

- $n = 6$: As $D_k^{1/d} < f_3(6, 3, 2.1) < 2.5$, from Proposition 2 we conclude that if $n = 6$, $d < 3$. If $d = 2$, $D_k^{1/d} < f_3(6, 2, 1.54) < 2.36$. Therefore, $D_k < 6$, which implies that $k = \mathbb{Q}(\sqrt{5})$ is the only possibility.
- $n = 5$: As $D_k^{1/d} < f_3(5, 3, 1.8) < 2.9$, from Proposition 2 we conclude that if $n = 5$, $d < 3$. If $d = 2$, $D_k^{1/d} < f_3(5, 2, 1.3) < 2.9$. Therefore, $D_k < 9$. So there are two possible real quadratic fields k , their discriminants are 5 and 8. Both the fields have class number 1, and we use the bound (21) to obtain $D_k^{1/d} < f_1(5, 2, 1) < 2.73$. So only $D_k = 5$ can occur.
- $n = 4$: As $D_k^{1/d} < f_3(4, 3, 1.3) < 3.61$, from Proposition 2 we conclude that if $n = 4$, $d < 3$. Let us assume that $d = 2$. Then since $D_k^{1/d} < f_3(4, 2, 1.1) < 3.76$, $D_k < 15$ and so the possible values of D_k are 5, 8, 12 or 13. The quadratic fields with these D_k have class number 1. Now from bound (21) we obtain $D_k^{1/d} < f_1(4, 2, 1) < 3.4$. Hence, $D_k < 12$, and only $D_k = 5, 8$ can occur.
- $n = 3$: As $D_k^{1/d} < f_3(3, 4, 1.3) < 5.1$, from Proposition 2 we conclude that if $n = 3$, $d < 4$. Now since $D_k^{1/d} < f_3(3, 3, 1.13) < 5.21$, if $d = 3 = n$, $D_k < 142$ from which we infer that $D_k = 49$ or 81. Since $D_k^{1/d} < f_3(3, 2, 0.8) < 5.6$, if $d = 2$ (and $n = 3$), $D_k < 32$, and in this case the possible values of D_k are 5, 8, 12, 13, 17, 21, 24, 28 or 29. The quadratic fields with these discriminants have class number 1, and we

use bound (21) to obtain $D_k^{1/d} < f_1(3, 2, 1) < 4.52$. Hence, $D_k < 21$ and only $D_k = 5, 8, 12, 13, 17$ can occur.

• $n = 2$: As $D_k^{1/d} < f_3(2, 7, 1.1) < 9$, Proposition 2 implies that $d \leq 6$.

$n = 2$ and $d = 6$: As $D_k^{1/d} < f_3(2, 6, 1) < 9$, $D_k < 531441$. One can check from the table in [1] that $h_k = 1$ for all the five number fields satisfying this bound. We now use bound (21) to obtain $D_k^{1/d} < f_1(2, 6, 1) < 7.2$, which contradicts Proposition 2.

$n = 2$ and $d = 5$: As $D_k^{1/d} < f_3(2, 5, 1) < 9.3$, $D_k < 69569$. Again, one can check from the table in [1] that there are five such number fields and the class number of each of them is 1. Now we use bound (21) to obtain $D_k^{1/d} < f_1(2, 5, 1) < 7.1$. Hence, $D_k < 18043$. From [1] we find that $D_k = 14641$ is the only possibility.

$n = 2$ and $d = 4$: As $D_k^{1/d} < f_3(2, 4, 0.92) < 9.74$, $D_k < 9000$. According to [1], there are 45 totally real quartic number fields with discriminant < 9000 , all of them have class number 1. We use bound (21) to obtain $D_k^{1/d} < f_1(2, 4, 1) < 7.037$. Hence, $D_k < 2453$. We find from [1] that there are eight totally real quartic number fields k with $D_k < 2453$. Their discriminants are

$$725, 1125, 1600, 1957, 2000, 2048, 2225, 2304.$$

$n = 2$ and $d = 3$: As $D_k^{1/d} < f_3(2, 3, 0.78) < 10.5$, $D_k < 1158$. From table B.4 of [C] we find that there are altogether 31 totally real cubics satisfying this discriminant bound. Each of these fields have class number 1. We use bound (21) to obtain $D_k^{1/d} < f_1(2, 3, 1) < 6.96$, which implies that $D_k < 338$. There are eight real cubic number fields satisfying this bound. The values of D_k are

$$49, 81, 148, 169, 229, 257, 316, 321.$$

$n = 2$ and $d = 2$: As $D_k^{1/d} < f_3(2, 2, 0.52) < 12$, $D_k < 144$. From table B.2 of totally real quadratic number fields given in [C], we check that the class number of all these fields are bounded from above by 2. Hence, $D_k^{1/d} < f_1(2, 2, 2) < 7.285$. So $D_k < 53$. Among the real quadratic fields with $D_k < 53$, there is only one field whose class number is 2, it is the field with $D_k = 40$. All the rest have class number 1, and from bound (21) we conclude that $D_k^{1/d} < f_1(2, 2, 1) < 6.8$, i.e., $D_k < 47$. Therefore, the following is the list of the possible values of D_k :

$$5, 8, 12, 13, 17, 21, 24, 28, 29, 33, 37, 40, 41, 44.$$

To summarize, for G of type B_n or C_n , the possible n , d and D_k are given in the following table.

n	d	D_k
$2, \dots, 13$	1	1
6	2	5
5	2	5
4	2	5, 8
3	3	49, 81
3	2	5, 8, 12, 13, 17
2	5	14641
2	4	725, 1125, 1600, 1957, 2000, 2048, 2225, 2304
2	3	49, 81, 148, 169, 229, 257, 316, 321
2	2	5, 8, 12, 13, 17, 21, 24, 28, 29, 33, 37, 40, 41, 44.

4.3. We will show that none of the possibilities listed in the above table actually give rise to an arithmetic fake compact hermitian symmetric space of type B_n or C_n . For this we recall first of all that \overline{G} , and so also G , are anisotropic over k (1.5). Now we observe that if G is a group of type B_n ($n \geq 2$), then it is k -isotropic if and only if it is isotropic at all real places of k (this is an immediate consequence of the classical Hasse principle for quadratic forms which says that a quadratic form over k is isotropic if and only if it is isotropic at every place of k , and the well-known fact that a quadratic form of dimension > 4 is isotropic at every nonarchimedean place). Also, a k -group of type C_n ($n \geq 2$) is isotropic if it is isotropic at all the real places of k (this is known, and follows, for example, from Proposition 7.1 of [PR]). These results imply that if $d = 1$, i.e., if $k = \mathbb{Q}$, then $G!$ is isotropic, and so $k = \mathbb{Q}$ is not possible.

Now let us take up the case where $d = 2$, i.e., k is a real quadratic field, and $n = 2, 5$ or 6 . Then for any real place v of k where G is isotropic, the complex dimension of the symmetric space of $G(k_v)$ is odd (recall from 1.4 that the complex dimension of the symmetric space of $G(k_v)$ is $2n - 1$ if G is of type B_n , and it is $n(n + 1)/2$ if G is of type C_n). But as the complex dimension of the hermitian symmetric space X is even (since the orbifold Euler-Poincaré characteristic of Γ is positive, see 1.3), we conclude that G must be isotropic at both the real places of k (note that G is anisotropic at a place v of k if and only if $G(k_v)$ is compact). From this observation we conclude that G is k -isotropic also in case $d = 2$, and $n = 2, 5$ or 6 . Therefore these possibilities do not occur.

Now we will rule out the case where $n = 2$, $d = 5$, and $D_k = 14641$. In this case, $k = \mathbb{Q}[x]/(x^5 - x^4 - 4x^3 + 3x^2 + 3x - 1)$. The class number of k is 1. It is easy to see that the cardinality q_v of the residue field of k at any nonarchimedean place v is at least 4.

Since $\mu(\mathcal{G}/\Gamma)$ is a submultiple of $1/2^r$ (2.1), and $[\Gamma : \Lambda] = 2^m$, where $m \leq 4 + r + \#\mathcal{J}$, see (15), using (3) we obtain

$$(24) \quad 1 \geq 2^r \mu(\mathcal{G}/\Gamma) = 2^r \frac{\mu(\mathcal{G}/\Lambda)}{[\Gamma : \Lambda]} > \frac{2^r}{[\Gamma : \Lambda]} \left(\frac{6D_k}{(2\pi)^6} \right)^5 \varepsilon > 2^{-4-\#\mathcal{J}} \left(\frac{87846}{(2\pi)^6} \right)^5 \prod_{v \in \mathcal{J}} e(P_v),$$

where $\mathcal{E} = \prod_{v \in V_f} e(P_v)$, with $e(P_v)$ as in 2.1.

The group G is a simply connected k -anisotropic group of type C_2 . Such a group is described in terms of a quaternion division algebra D , with center k (see [Ti1]). Since the symmetric space X is a complex analytic space of even complex dimension, and $d = 5$, we conclude that G is anisotropic at an odd number of real places of k . Hence the quaternion division algebra D ramifies at an odd number of real places of k . Since a quaternion division algebra ramifies at an even number of places, we infer that there is at least one nonarchimedean place where D ramifies. At such a place, G is of rank 1, so such a place lies in \mathcal{T} . This shows that \mathcal{T} is nonempty. For $v \in \mathcal{T}$, $e(P_v)$ equals either $q_v^6/(q_v^4 - 1)$, or $q_v^6/(q_v + 1)(q_v^2 - 1)$, or $q_v^6/(q_v^2 - 1)$. All these numbers are larger than q_v^2 . Now as $q_v \geq 4$, and \mathcal{T} is nonempty, from (24) we conclude that

$$1 \geq 2^r \mu(G/\Gamma) > 2^{-4-\#\mathcal{T}} \left(\frac{87846}{(2\pi)^6} \right)^5 16^{\#\mathcal{T}} > \frac{1}{2} \left(\frac{87846}{(2\pi)^6} \right)^5 > 1,$$

which is absurd.

4.4. To rule out the remaining cases listed in the table in 4.2, we compute the value of \mathcal{R} (\mathcal{R} as in (12)) in each case. These values are given below. It turns out that in none of the remaining cases the numerator of \mathcal{R} is a power of 2 and Proposition 1 then eliminates these cases.

n	d	k	D_k	$\zeta_k(-1)$	$\zeta_k(-3)$	$\zeta_k(-5)$	$\zeta(-7)$	\mathcal{R}
4	2	$x^2 - 5$	5	1/30	1/60	67/630	361/120	24187/34836480000
4	2	$x^2 - 2$	8	1/12	11/120	361/252	24611/240	97730281/22295347200.

n	d	k	D_k	$\zeta_k(-1)$	$\zeta_k(-3)$	$\zeta_k(-5)$	\mathcal{R}
3	3	$x^3 - x^2 - 2x + 1$	49	-1/21	79/210	-7393/63	584047/142248960
3	3	$x^3 - 3x - 1$	81	-1/9	199/90	-50353/27	10020247/11197440
3	2	$x^2 - 17$	17	1/3	41/30	5791/63	237431/362880
3	2	$x^2 - 13$	13	1/6	29/60	33463/1638	970427/37739520
3	2	$x^2 - 3$	12	1/6	23/60	1681/126	38663/2903040
3	2	$x^2 - 2$	8	1/12	11/120	361/252	3971/23224320
3	2	$x^2 - 5$	5	1/30	1/60	67/630	67/72576000

n	d	k	D_k	$\zeta_k(-1)$	$\zeta_k(-3)$	\mathcal{R}
2	4	$x^4 - 4x^2 + 1$	2304	1	22011/10	22011/2560
2	4	$x^4 - x^3 - 5x^2 + 2x + 4$	2304	4/5	9202/5	4601/800
2	4	$x^4 - 4x^2 + 2$	2048	5/6	87439/60	87439/18432
2	4	$x^4 - 5x^2 + 5$	2000	2/3	3793/3	3793/1152
2	4	$x^4 - 4x^2 - x + 1$	1957	2/3	3541/3	3541/1152
2	4	$x^4 - 6x^2 + 4$	1600	7/15	17347/30	121429/115200
2	4	$x^4 - x^3 - 4x^2 + 4x + 1$	1125	4/15	2522/15	1261/7200
2	4	$x^4 - x^3 - 3x^2 + x + 1$	725	2/15	541/15	541/28800
2	3	$x^3 - x^2 - 4x + 1$	321	-1	555/2	555/128
2	3	$x^3 - x^2 - 4x + 2$	316	-4/3	874/3	437/72
2	3	$x^3 - x^2 - 4x + 3$	257	-2/3	1891/15	1891/1440
2	3	$x^3 - 4x - 1$	229	-2/3	1333/15	1333/1440
2	3	$x^3 - x^2 - 4x - 1$	169	-1/3	11227/390	11227/74880
2	3	$x^3 - x^2 - 3x + 1$	148	-1/3	577/30	577/5760
2	3	$x^3 - 3x - 1$	81	-1/9	199/90	199/51840
2	3	$x^3 - x^2 - 2x + 1$	49	1/21	79/210	79/282240

5. G of type D_n

We will consider hermitian symmetric spaces associated to Lie group of type D_n , with $n \geq 4$. Note that a group of type D_n , with $n < 4$, is also of type A_n , and hermitian locally symmetric spaces of type A_n have already been studied in [PY2] and [PY3]. The compact hermitian symmetric spaces are of the form $SO^*(2n)/U(n)$ or $SO(2, 2n-2)/S(O(2) \times O(2n-2))$. In the terminology of Élie Cartan, these are hermitian symmetric spaces of type IV or type II.

We note that any absolutely simple algebraic group G over \mathbb{Q} of type 1D_n and 2D_n , with $n \geq 4$, whose real rank is at least 2, is \mathbb{Q} -isotropic, and hence, by Godement criterion, its arithmetic subgroups are non-cocompact in $G(\mathbb{R})$. Since we are only interested in compact hermitian locally symmetric spaces (and $SO(2, 2n-2)$ is of real rank 2, and for $n \geq 4$, the real rank of $SO^*(2n)$ is at least 2) in this paper the number field k will always be a nontrivial extension of \mathbb{Q} .

Let $\mathcal{G} = \prod_{j=1}^r G(k_{v_j})$, and let $\tilde{\Pi}$ be the inverse image of Π in \mathcal{G} . Then the kernel of the homomorphism $\pi : \mathcal{G} \rightarrow \bar{\mathcal{G}}$ is of order 4^r , and hence the orbifold Euler-Poincaré characteristic $\chi(\tilde{\Pi})$ of $\tilde{\Pi}$ equals $\chi(\Pi)/4^r = \chi(X_u)/4^r$. Let \mathcal{T} be the set of nonarchimedean places v of k which are unramified in ℓ and P_v is not a hyperspecial parahoric subgroup of $G(k_v)$.

The expression can be rewritten in form

$$(25) \quad \mu(\mathcal{G}/\Lambda) = \mathcal{R} \prod_{v \in \mathcal{T} \cup \mathcal{T}'} e'(P_v),$$

where

$$(26) \quad \mathfrak{R} = 2^{-dn} \left| \prod_{j=1}^n Z(-m_j) \right|,$$

where $Z(s)$ is either $\zeta_k(s)$ or $L_{\ell|k}(s)$ according as s is odd or even, and whether ℓ is a totally real or complex extension of k .

5.1. The relevant set of indices in the formula of Prasad [P] is similar to the case of type II, namely, $m_i = 1, 3, 5, \dots, 2n - 3$, together with $n - 1$ with multiplicity two if n is even and multiplicity 1 if n is odd. $\mathfrak{s} = 2n - 1$. The center of G is of order 4. $\dim G = rn(2n - 1)$. We have the following three cases. In the following, τ is 2 if n is even, and τ is 4 if n is odd. Clearly, $h_{k,\tau} \leq h_{k,2} \leq h_k$ and similarly for $h_{\ell,\tau}$.

Case (a): n is even, and G is an inner k -form of a split group. Then

$$[\Gamma : \Lambda] \leq h_{k,\tau}^2 \cdot 2^{2(d+r-1+\#\mathcal{T})}.$$

Case (b): n is even, G is of type 2D_n . Then ℓ is a totally real quadratic extension of k , and

$$[\Gamma : \Lambda] \leq h_{\ell,\tau} \cdot 2^{2(d+r+\#\mathcal{T})-1} D_{\ell} / D_k^2.$$

Case (c): n is odd, then ℓ is a totally complex quadratic extension of k , and

$$[\Gamma : \Lambda] \leq h_{\ell,\tau} \cdot 2^{2(d+r+\#\mathcal{T})}.$$

Case (d): $n = 4$, triality forms ${}^{3,6}D_4$, ℓ is a totally real cubic extension of k such that over the normal closure of ℓ/k , G is an inner form of a split group.

$$[\Gamma : \Lambda] \leq h_{\ell,2} \cdot 2^{2(d+r+\#\mathcal{T})} D_{\ell} / D_k^3.$$

Case (a)

5.3. In this case, $n > 2$ is even, with exponents m_j given by 5.2?. The corresponding expression for \mathfrak{R} is given by

$$\mathfrak{R} := 2^{-nd} \left| \left(\prod_{j=1}^{n-1} \zeta_k(1 - 2j) \right) \zeta_k(1 - n) \right|$$

The relevant volume formula is

$$(27) \quad \mu(G(k_{v_o})/\Lambda) = D_k^{n(2n-1)/2} \left(\left\{ \frac{(n-1)!}{(2\pi)^n} \right\} \prod_{j=1}^{n-1} \frac{(2j-1)!}{(2\pi)^{2j}} \right)^d \mathcal{E}.$$

Letting $A(n) = \left\{ \frac{(2\pi)^n}{(n-1)!} \right\} \prod_{j=1}^{n-1} \frac{(2\pi)^{2j}}{(2j-1)!}$ and using the estimates that $[\Gamma : \Lambda] \leq h_{k,2}^2 \cdot 2^{2(d+r+\#\mathcal{T}-1)}$, $\mu(G(k_{v_o})/\Lambda) \leq 1/4^r$ and the fact that $\mathcal{E} = \prod_{\mathcal{E}\mathcal{T}} e(P_v) > 4^{\#\mathcal{T}}$, we conclude from equation (27) that

$$(28) \quad D_k^{1/d} \leq a_1(n, d, h_{k,2}) := \left[\{4A(n)\}^d \cdot \frac{h_{k,2}^2}{4} \right]^{2/dn(2n-1)}.$$

Recall Brauer-Siegel Theorem and Zimmert's estimates for totally real field k of degree d ,

$$\begin{aligned} h_k &\leq \frac{w_k}{R_k} s(s-1) 2^{-d} \Gamma(s/2)^d (\pi^{-d} D_k)^{s/2} \zeta_k(s) \\ \frac{R_k}{w_k} &\geq 0.02 e^{0.46d} \end{aligned}$$

we get

$$(29) \quad \begin{aligned} D_k^{1/d} &< a_3(n, d, \delta), \\ &:= \left[\left\{ \frac{[\Gamma((1+\delta)/2)\zeta(1+\delta)]^2}{(\pi)^{1+\delta} e^{0.92}} A(n) \right\} \cdot \left\{ \frac{25}{2} \delta(1+\delta) \right\}^{2/d} \right]^{2/(2n^2-n-2-2\delta)}. \end{aligned}$$

For fixed n and δ , $a_3(n, d, \delta)$ clearly decreases as d increases. Furthermore, for a given d and δ , $a_3(n, d, \delta)$ decreases as n increases provided $n \geq 14$.

$$\begin{aligned} D_k^{1/d} &< a_1(n, d, h_{k,2}) := \left[\{4A(n)\}^d \cdot \frac{h_{k,2}^2}{2} \right]^{2/dn(2n-1)} \\ D_k^{1/d} &< a_3(n, d, \delta), \\ &:= \left[\left\{ \frac{[\Gamma((1+\delta)/2)\zeta(1+\delta)]^2}{(\pi)^{1+\delta} e^{0.92}} A(n) \right\} \cdot \{25\delta(1+\delta)\}^{2/d} \right]^{2/(2n^2-n-2-2\delta)} \end{aligned}$$

We obtain by a direct computation the following upper bound for the value of $a_3(n, 2, 3)$ for small n .

$$\begin{array}{cccccc} n & 4 & 6 & 8 & 10 & 12 \\ a_3(n, 2, 4) & < 7.3 & 2.95 & 2 & 1.54 & 1.26 \end{array}$$

We conclude that $k = \mathbb{Q}$ for all even $n \geq 8$. But as $k \neq \mathbb{Q}$, see the introduction, $n \leq 6$.

Consider now the case that $n = 6$. Assume now that $d \geq 3$, $D_k^{1/d} \leq a_3(6, 3, 2.22) < 2.78$. Hence from Proposition 1, $d < 3$.

For $d = 2$, $D_k^{1/d} \leq a_3(6, 2, 1.96) < 2.85$. Since $2.85^2 < 8.2$, $D_k = 5$, or 8. As the class number is 1 for the four fields, we know that $D_k^{1/d} \leq a_1(6, 2, 1) < 2.82$. As $2.82^2 < 7.96$, we conclude that $D_k = 5$.

Consider now $n = 4$. Observe that for $d \geq 4$, $D_k^{1/d} \leq a_3(4, 4, 1.559) < 5.1$. which contradicts the lower bound of root discriminant for $d = 4$ in Proposition 1. Hence $d < 4$.

For $d = 3$, $D_k^{1/d} \leq a_3(4, 3, 1.45) < 5.2$. As $5.2^3 < 141$. We refer to the table of totally real cubics in [1] that $h_k = 1$ for the number fields satisfying the above bound. Hence $D_k^{1/d} \leq a_1(4, 3, 1) < 4.8$. We conclude that $D_k < 4.8^3 < 111$. Hence D_k can only take one of the following values,

$$49, 81.$$

For $d = 2$, $D_k^{1/d} \leq a_3(4, 2, 1.26) < 5.47$. As $5.47^2 < 30$. We refer to the table of real quadratics as in [C] that $h_k = 1$. Hence $D_k^{1/d} \leq a_1(4, 2, 1) < 4.73$. As $4.73^2 < 23$,

we conclude that D_k can only take one of the following values,

$$5, 8, 12, 13, 17, 21$$

Here is a summary of all the possibilities for the totally real number field k apart from $k = \mathbb{Q}$.

n	d	D_k
6	2	5
4	3	49, 81
4	2	5, 8, 12, 13, 17, 21

The following is the summary of for $n = 4$ according to the above table., where $k = \mathbb{Q}(\sqrt{a})$.

n	d	D_k	$\zeta_k(-1)$	$\zeta_k(-3)$	$\zeta_k(-5)$	\mathcal{R}
4	3	49	$-1/21$	$79/210$	$-7393/63$	46139713/238978252800
4	3	81	$-1/9$	$199/90$	$-50353/27$	1994029153/8062156800
4	2	3	$1/6$	$23/60$	$1681/126$	889249/696729600
4	2	5	$1/30$	$1/60$	$354497/33333330$	354497/921599907840000
4	2	8	$1/12$	$825/9000$	$361/252$	43681/11147673600
4	2	13	$1/6$	$29/60$	$33463/1638$	28142383/9057484800
4	2	17	$1/3$	$41/30$	$5791/63$	9734671/43545600
4	2	21	$1/3$	$77/30$	$17971/63$	15221437/6220800

Calculation of \mathcal{R} for the case of $n = 6$ is listed in the following table.

n	a	$\zeta_k(-1)$	$\zeta_k(-3)$	$\zeta_k(-5)$	$\zeta_k(-7)$	$\zeta_k(-9)$	\mathcal{R}
6	5	$1/30$	$1/60$	$67/630$	$361/120$	$412751/1650$	668874965279/57940033536000000
6	2	$1/12$	$11/120$	$361/252$	$24611/240$	$2873041/132$	9214790966898371/1078738078924800

From the last column, clearly none of them are candidates for an arithmetic fake compact hermitian space according to Proposition 1.

Case (b)

5.4. In this case, n is an even number. With the values of m_i are given by 5.3?, the expression for \mathcal{R} is given by

$$\mathcal{R} := 2^{-nd} \left| \left(\prod_{j=1}^{n-1} \zeta_k(1-2j) \right) L_{\ell|k}(1-n) \right|$$

The relevant volume formula is

$$(30) \quad \mu(G(k_{v_o})/\Lambda) = D_k^{n(2n-1)/2} (D_\ell/D_k^2)^{(2n-1)/2} \left(\left\{ \frac{(n-1)!}{(2\pi)^n} \right\} \prod_{j=1}^{n-1} \frac{(2j-1)!}{(2\pi)^{2j}} \right)^d \mathcal{E}.$$

Letting $B(n) = \left\{ \frac{(2\pi)^n}{(n-1)!} \right\} \prod_{j=1}^{n-1} \frac{(2\pi)^{2j}}{(2j-1)!}$ and using the estimates that $[\Gamma : \Lambda] \leq h_{\ell,2} \cdot 2^{2(d+r+\#\mathcal{J})-1} D_\ell/D_k^2$, $\mu(G(k_{v_o})/\Gamma) \leq 1/4^r$, and the fact that $\mathcal{E} = \prod_{\epsilon \in \mathcal{J}} e(P_\nu) > 4^{\#\mathcal{J}}$, we conclude from equation (30) that

$$\begin{aligned}
D_k^{1/d} &< b_2(n, d, h_{\ell,2}) := [\{4B(n)\}^d \cdot \frac{h_{\ell,2}}{2}]^{2/dn(2n-1)}, \\
D_k^{1/d} &< b_3(n, d, \delta), \\
&:= \left[\left\{ \frac{[\Gamma((1+\delta)/2)\zeta(1+\delta)]^2}{(\pi)^{1+\delta} e^{0.92}} B(n) \right\} \cdot \{25\delta(1+\delta)\}^{1/d} \right]^{2/(2n^2-n-2-2\delta)}, \\
D_\ell/D_k^2 &< \mathfrak{r}_1(n, d, D_k, h_{\ell,2}) := \left(\frac{h_{\ell,2}}{2} (4B(n))^d D_k^{-n(2n-1)/2} \right)^{2/(2n-3)}, \\
D_\ell/D_k^2 &< \mathfrak{r}_2(n, d, D_k, R_\ell/w_\ell, \delta), \\
&:= \left(\frac{\delta(\delta+1)}{2R_\ell/w_\ell} D_k^{-(n(2n-1)-2-2\delta)/2} \left\{ B(n) \frac{[\Gamma(\frac{1+\delta}{2})\zeta(1+\delta)]^2}{\pi^{(1+\delta)}} \right\}^d \right)^{2/(2n-4-\delta)}, \\
D_\ell/D_k^2 &< \mathfrak{r}_3(n, d, D_k, \delta), \\
&:= \left(25\delta(\delta+1) D_k^{-(n(2n-1)-2-2\delta)/2} \left\{ B(n) \frac{[\Gamma(\frac{1+\delta}{2})\zeta(1+\delta)]^2}{\pi^{(1+\delta)} e^{0.92}} \right\}^d \right)^{2/(2n-4-\delta)}, \\
D_\ell^{1/2d} &< \mathfrak{t}_1(n, d, D_k, h_{\ell,2}) := (\mathfrak{r}_1(n, d, D_k, h_{\ell,2}) D_k^2)^{1/2d} \\
D_\ell^{1/2d} &< \mathfrak{t}_2(n, d, D_k, R_\ell/w_\ell, \delta) := (\mathfrak{r}_2(n, d, D_k, R_\ell/w_\ell, \delta) D_k^2)^{1/2d} \\
D_\ell^{1/2d} &< \mathfrak{t}_3(n, d, D_k, \delta) := (\mathfrak{r}_3(n, d, D_k, \delta) D_k^2)^{1/2d}
\end{aligned}$$

5.5.

Lemma 1. *Let $\delta \in [0.004, 4]$. For fixed values of n and δ , $f_3(n, d, \delta)$ decreases as d increases. Furthermore, for fixed values of d and δ , if $n \geq 8$, then $f_3(n+1, d, \delta) < \max(1, f_3(n+1, d, \delta))$.*

Proof It is obvious that for fixed n and $\delta \in [0.04, 9]$, $f_3(n, d, \delta)$ decreases as d increases. Now we observe that for $n \geq 9$, $(2n-1)! > (2\pi)^{2n}$. From this it is easy to see that if for a given d, δ , and $n \geq 8$, $f_3(n, d, \delta) \geq 1$, then $f_3(n+1, d, \delta) < f_3(n, d, \delta)$, and if $f_3(n, d, \delta) < 1$, then $f_3(n+1, d, \delta) < 1$. In particular, if for given d , and $\delta \in [0.04, 9]$, $f_3(8, d, \delta) < c$, with $c \geq 1$, then $f_3(n, d', \delta) < c$ for all $n \geq 8$ and $d' \geq d$.

5.6. We obtain by a direct computation the following upper bound for the value of $b_3(n, 2, 3)$ for small n .

$$\begin{array}{cccc}
n & 4 & 6 & 8 \\
b_3(n, 2, 4) & < & 7.3 & 3 & 2.
\end{array}$$

From Proposition 1 and the fact that b_3 decreases as n increases, we conclude that $k = \mathbb{Q}$ for even $n \geq 8$. But $k \neq \mathbb{Q}$, see the introduction, and hence $n \leq 6$.

Consider now $n = 6$. As $b_3(6, 3, 2.2) < 2.8$, which contradicts the lower bound in Proposition 1. Hence $d < 3$.

For $d = 2$, $D_k^{1/d} < b_3(6, 2, 1.96) < 2.85$. Since $2.85^2 < 9$, $D_k = 5, 8$ are the only possibilities.

Consider now the case that $D_k = 8$. From bounds in **5.4** that $D_\ell^{1/2d} \leq t_3(6, 2, 8, 1.93) < 2.91$. This contradicts the lower bound of Proposition 2.

Consider now the case that $D_k = 5$. Hence from estimates in **3.1** $D_\ell^{1/2d} \leq t_3(6, 2, 5, 1.27) < 7.28$. From Proposition 1 and also Table IV of [Ma], $N_r(6) > 7.9$. Hence by considering Hilbert Class Field, $h_\ell \leq \lfloor 4/4 \rfloor = 1$. It follows that $D_\ell^{1/2d} \leq t_1(6, 2, 5, 1) < 4.47$. This contradicts Proposition 2 again. Hence it cannot happen that $\deg(k) > 1$ for $n = 6$.

Consider now $n = 4$. From bounds in **4.4**, for $d = 4$, $D_k^{1/d} < b_3(4, 4, 1.56) < 5.05$, which contradicts Proposition 2. It follows from the decreasing properties of f_3 with respect to d for appropriate fixed n and d that $d \leq 3$.

For $d = 3$, we know from Proposition 1 that $D_k \geq 49$. Hence from estimates in **3.1** $D_\ell^{1/2d} \leq t_3(4, 3, 49, 0.87) < 16.5$ From Table IV of [Ma], $N_r(14) > 17$. Hence by considering Hilbert Class Field, $h_\ell \leq \lfloor 13/6 \rfloor = 2$. It follows that $D_\ell^{1/2d} \leq t_1(4, 3, 49, 2) < 8.2$. From Table IV of [Ma] again, $N_r(7) > 9$. Hence $h_\ell \leq \lfloor 6/6 \rfloor = 1$. It follows that $D_\ell^{1/2d} \leq t_1(4, 3, 49, 1) < 7.9$. This contradicts the lower bound of root discriminant on totally real number field of degree 6 given by Proposition 2.

For $d = 2$, we know that $D_k \geq 8$ except from 5. Consider first the cases of $D_k \geq 8$. From bounds in **5.4** $D_\ell^{1/2d} \leq t_3(4, 2, 8, 0.54) < 39.1$. From Table IV of [Ma], $N_r(80) > 39.4$. Hence by considering Hilbert Class Field, $h_\ell \leq \lfloor 79/4 \rfloor = 19$. Hence $h_{\ell,2} \leq 16$. It follows that $D_\ell^{1/2d} \leq t_1(4, 2, 8, 16) < 15.67$. From Table IV of [Ma], $N_r(13) > 16$. Hence by considering Hilbert Class Field, $h_\ell \leq 12/4 = 3$. Hence $h_{\ell,2} = 2$. It follows that $D_\ell^{1/2d} \leq t_1(4, 2, 8, 2) < 12.724$. It follows that $D_\ell \leq 26211$. The totally real quartics with discriminant bound as above are contained in table t44.001 and t44.002 of [1]. where the number fields all have $h \leq 3$. Hence $h_{\ell,2} \leq 2$. We now compute that $D_\ell^{1/4} \leq t_1(4, 2, 8, 2) < 12.724$. Hence $D_\ell \leq 26211$ in cases that $D_k \geq 8$.

Suppose now we consider $D_k \geq 12$. The previous discussions imply that $h_{\ell,2} = 2$. We compute that $D_\ell^{1/2d} \leq t_1(4, 2, 12, 2) < 8.83$. From Table IV of [Ma], $N_r(7) > 9.3$. Hence by considering Hilbert Class Field, $h_\ell \leq \lfloor 6/4 \rfloor = 1$. It follows that $D_\ell^{1/2d} \leq t_1(4, 2, 12, 1) < 8.242$. Hence $D_\ell \leq 4614$.

From t44.001 again, we check that there are only 12 such totally real quartics, with D_ℓ given by

725, 1125, 1600, 1957, 2000, 2048, 2225, 2304, 2525, 2624, 2777, 3600, 3981, 4205, 4352, 4400, 4525.

Furthermore, with our assumption of $D_\ell \geq 12$ $D_k^{1/2} \leq b_1(4, 2, 1) < 4.73$. In particular, D_k can only take the values of 12, 13, 17, 21 according to the list of totally real quadrics. Since D_ℓ has to be an integral multiple of D_k^2 , we check easily that for $D_k \geq 12$, the only compatible set of (D_k, D_ℓ) are given by (12, 2304) and (12, 3600).

For $D_k = 8$, we have shown that $D_\ell \leq 26211$.

Consider now the case of $D_k = 5$. We claim that $D_\ell^{1/2d} \leq 57$. Assume on the contrary that $D_\ell^{1/2d} > 57$. According to Remak's estimate as stated in (3.15) of [F],

we conclude that the regulator of ℓ has the following estimates,

$$R_\ell \geq \frac{\log D_\ell - d \log d}{(2^{1/3} d^{1/(d-1)} (d^3-d)/3)^{1/2}},$$

where $d = 2$. Hence as $D_\ell > 57^4$, we conclude that $R_\ell \geq 4.744$. Since $w_\ell = 2$, we obtain from the bounds in **5.4** that $D_\ell^{1/4} \leq \mathfrak{t}_1(4, 2, 5, 4.744/2, 0.5) < 47.1$, contradicting the assumption that $D_\ell^{1/2d} > 57$. Hence the claim is proved.

As $D_\ell^{1/2d} \leq 57$, we observe from Table IV of [Ma], $N_r(81500) > 57$. Hence by considering Hilbert Class Field, $h_\ell \leq \lfloor 1499/4 \rfloor = 374$. It follows that $D_\ell^{1/2d} \leq \mathfrak{t}_1(4, 2, 5, 374) < 32.78$. From Table IV of [Ma], $N_r(46) > 32.8$. Hence by considering Hilbert Class Field, $h_\ell \leq \lfloor 45/4 \rfloor = 11$. Hence $h_{\ell,2} \leq 8$. It follows that $D_\ell^{1/2d} \leq \mathfrak{t}_1(4, 2, 5, 8) < 22.32$. Again, from Table IV of [Ma], $N_r(21) > 22.38$. Hence by considering Hilbert Class Field, $h_\ell \leq \lfloor 20/4 \rfloor = 5$. Hence $h_{\ell,2} \leq 4$. It follows that $D_\ell^{1/2d} \leq \mathfrak{t}_1(4, 2, 5, 4) < 20.82$. Hence $D_\ell \leq 187898$. According to table t44001-t44002 of [1], $h_\ell \leq 3$ and $h_{\ell,2} \leq 2$. It follows that $D_\ell \leq \lfloor \mathfrak{t}_1(4, 2, 5, 2)^4 \rfloor \leq 142318$.

Here is a summary of all the possibilities:

n	d	D_k	D_ℓ
4	2	12	2304, 3600
4	2	8	≤ 26211
4	2	5	≤ 142318

5.7. Malle has provided us the list of number fields satisfying the above constraints. For $k = \sqrt{5}$, there are 273 number fields ℓ obtained from a totally real quadratic extension satisfying the above constraint. Similarly for $k = \sqrt{8}$, there are 28 such number fields ℓ . For each of the above case as well as the two cases with $(D_k, D_\ell) = (12, 2304)$ and $(12, 3600)$, we compute the value of \mathcal{R} . It turns out that they all violate the Proposition 1 in the sense that the numerators are not powers of $s = 2$. Hence we conclude that there is no fake compact hermitian symmetric space of type D_n for $n > 4$.

Case (c)

5.8. We refer to **1.3** for the relevant set of m_i . The expression for \mathcal{R} is given by

$$\mathcal{R} := 2^{-nd} \left| \left(\prod_{j=1}^{n-1} \zeta_k(1-2j) \right) L_{\ell|k}(1-n) \right|$$

The relevant volume formula is

$$(31) \quad \mu(G(k_{v_o})/\Lambda) = D_k^{n(2n-1)/2} (D_\ell/D_k^2)^{(2n-1)/2} \left(\left\{ \frac{(n-1)!}{(2\pi)^n} \right\} \prod_{j=1}^{n-1} \frac{(2j-1)!}{(2\pi)^{2j}} \right)^d \mathcal{E}.$$

Letting $C(n) = \left\{ \frac{(2\pi)^n}{(n-1)!} \right\} \prod_{j=1}^{n-1} \frac{(2\pi)^{2j}}{(2j-1)!}$ and using the estimates that $[\Gamma : \Lambda] \leq h_{\ell,2} \cdot 2^{2(d+r+\#\mathcal{J})}$, $\mu(G(k_{v_o})/\Lambda) < 1/4^r$ and the fact that $\mathcal{E} = \prod_{v \in \mathcal{J}} e(P_v) > 4^{\#\mathcal{J}}$, we conclude from equation (3) again that

$$\begin{aligned}
D_k^{1/d} &< l_1(n, d, h_{\ell,2}) := [\{4C(n)\}^d \cdot h_{\ell,2}]^{2/dn(2n-1)} \\
D_k^{1/d} &< l_3(n, d, \delta), \\
&:= \left[\left\{ \frac{[\Gamma(1+\delta)\zeta(1+\delta)]^2}{(2\pi)^{1+\delta} e^{0.1}} 4C(n) \right\} \cdot \{50\delta(1+\delta)\}^{1/d} \right]^{2/(2n^2-n-2-2\delta)} \\
D_\ell/D_k^2 &< t_1(n, d, D_k, h_{\ell,2}), \\
&:= (h_{\ell,2}(4C(n))^d D_k^{n(2n-1)/2})^{2/(2n-1)}. \\
D_\ell/D_k^2 &< t_2(n, d, D_k, R_\ell/w_\ell, \delta), \\
&:= \left(\frac{\delta(\delta+1)}{R_\ell/w_\ell} D_k^{-(n(2n-1)-2-2\delta)/2} \{4C(n) \frac{[\Gamma(\frac{1+\delta}{2})\zeta(1+\delta)]^2}{(\pi)^{(1+\delta)}}\}^d \right)^{2/(2n-2-\delta)}, \\
D_\ell/D_k^2 &< t_3(n, d, D_k, \delta), \\
&:= (50\delta(\delta+1) D_k^{-(n(2n-1)-2-2\delta)/2} \{4C(n) \frac{[\Gamma(1+\delta)\zeta(1+\delta)]^2}{(2\pi)^{(1+\delta)} e^{0.1}}\}^d)^{2/(2n-2-\delta)}; \\
D_\ell^{1/2d} &< u_1(n, d, D_k, h_{\ell,2}) := (t_1(n, d, D_k, h_{\ell,2}) D_k^2)^{1/2d} \\
D_\ell^{1/2d} &< u_2(n, d, D_k, R_\ell/w_\ell, \delta) := (t_2(n, d, D_k, R_\ell/w_\ell, \delta) D_k^2)^{1/2d} \\
D_\ell^{1/2d} &< u_3(n, d, D_k, \delta) := (t_3(n, d, D_k, \delta) D_k^2)^{1/2d}
\end{aligned}$$

In the above, we have applied Brauer-Siegel Theorem for a totally complex number field ℓ , which states that for all real $s > 1$,

$$(32) \quad h_\ell R_\ell \leq w_\ell s(s-1) \Gamma(s)^d ((2\pi)^{-2d} D_\ell)^{s/2} \zeta_\ell(s),$$

where h_ℓ is the class number and R_ℓ is the regulator of ℓ , and w_ℓ is the order of the finite group of roots of unity contained in ℓ . Furthermore, we have used the estimates $R_\ell \geq 0.02w_\ell e^{0.1d}$ of R. Zimmert [Z] for a totally complex number field.

5.9. We obtain by a direct computation the following upper bound for the value of $l_3(n, 2, 3)$ for small n .

$$\begin{array}{cccccc}
n & 3 & 5 & 7 & 9 \\
l_3(n, 2, 1.8) & < & 10.7 & 3.8 & 2.4 & 1.77
\end{array}$$

A simple calculations as in the proof of Lemma 1 shows that the conclusion of Lemma 1 is still applicable to Case (c). Hence from Proposition 2 and the fact that l_3 is decreasing in n , we conclude that $k = \mathbb{Q}$ for odd $n \geq 9$.

Consider now $n = 7$. Assume first that $d \geq 3$. Then $D_k^{1/d} \leq l_3(7, 3, 1.67) < 2.35$, contradicting Proposition 1. Hence $d \leq 2$.

In the case of $d = 2$, as $D_k \geq 8$ apart from 5, $D_\ell^{1/2d} \leq t_3(7, 2, 8, 1.51) < 1$ which again has no solution from Proposition 1.

For $D_k = 5$, $D_\ell^{1/2d} \leq t_3(7, 2, 5, 1.1) \leq 4.223$. Hence according to the table of number fields in t40.001 of [1], $D_\ell \leq 318$ and all the number fields have $h_\ell = 1$. It

follows that $D_\ell^{1/2d} \leq t_1(7, 2, 5, 1) < 2.79$, which contradicts Proposition 2 and hence there is no solution.

Consider now the case $n = 5$.

In this case, as $l_3(5, 3, 1.51) < 3.61$, which contradicts the lower bound in Proposition 1. Hence $d < 3$.

For $d = 2$, we know from Proposition 1 that $D_k \geq 5$. Hence from bounds in 4.1 $D_\ell^{1/2d} \leq u_3(5, 2, 5, 0.67) < 12.2$. From Table IV of [Ma], $N_c(34) > 12.2$. Hence by considering Hilbert Class Field, $h_\ell \leq \lfloor 33/4 \rfloor = 8$. It follows that $h_{\ell,4} \leq 4$ and $D_\ell^{1/2d} \leq u_1(5, 2, 5, 4) < 8.15$. From Table IV of [Ma], $N_c(16) > 8.7$. Hence by considering Hilbert Class Field, $h_\ell \leq \lfloor 15/4 \rfloor = 3$. It follows that $h_{\ell,4} = 1$ and $D_\ell^{1/2d} \leq u_1(5, 2, 5, 1) < 7.55$.

To obtain upper bound on D_k , we obtain from the estimates in 4.1 that $D_k^{1/d} < l_1(5, 2, 4) < 3.7$. Hence $D_k = 5, 8, 12$ or 13 .

Suppose $D_k = 13$, As $t_1(5, 2, 13, 1) < 1.1$, the only possibility for ℓ is $D_\ell = 169$. However, there is no totally complex quartic field with discriminant 169. Hence D_k cannot be 13.

Suppose $D_k = 12$, As $t_1(5, 2, 12, 1) < 1.7$, the only possibility for ℓ is $D_\ell = 144$, corresponding to $h_\ell = 1$.

Suppose $D_k = 8$. As $t_1(5, 2, 8, 1) < 12.36$. It follows that $D_\ell \leq 12 \times 8^2 \leq 768$. The number fields are listed in table t40.001 of [1]. Among these number fields, only $D_\ell = 256, 320, 512, 576, 592$ and 605 are allowed, where $D_\ell = 576$ correspond to two different number fields ℓ . Only the first four values are divisible by 8^2 . Hence D_ℓ can only take the values of $256, 320, 512, 576$.

Suppose $D_k = 5$, As $t_1(5, 2, 5, 1) < 130$, we conclude that $D_\ell \leq 5^2 \times 129 = 3225$. According to the table in [1], the only possibilities are

$$D_\ell = 125, 225, 400, 1025, 1225, 1525, 1600, 1825, 2725, 2925, 3025.$$

5.10. Among the pairs of (D_k, D_ℓ) obtained above, only some of them can be discriminants of some number fields k and ℓ such that ℓ is a totally complex quadratic extension of k . We eliminate the rest. In conclusion, here are all the possibilities for Case (c), where ℓ is a totally imaginary quadratic extension of a totally real number field k . The list of such number fields was provided by Malle.

n	d	D_k	D_ℓ
5	2	12	144
5	2	8	256, 576
5	2	5	125, 225, 400, 1025, 1225, 1525, 1600, 2725, 3025

5.11. The above list allows us to compute \mathcal{R} explicitly so that we may apply Proposition 1. It turns out that after computation that none of the remaining cases satisfy the conditions given by Proposition 1, as given by the following table, where $n = 5$. Hence Case (c) never occurs as well.

D_k	D_ℓ	$\zeta_k(-1)$	$\zeta_k(-3)$	$\zeta_k(-5)$	$\zeta_k(-7)$	$L_{\ell k}(-4)$	\mathcal{R}
12	144	1/6	23/60	1681/126	257543/120	5/3	9957385009/3344302080
8	256	1/12	11/120	361/252	24611/240	285/2	1856875339/11890851840
8	576	1/12	11/120	361/252	24611/240	15940/3	77891033957/13377208320
5	125	1/30	1/60	67/630	361/120	1172/25	7086791/870912000000
5	225	1/30	1/60	67/630	361/120	1984/3	749797/6531840000
5	400	1/30	1/60	67/630	361/120	8805	14197769/9289728000
5	1025	1/30	1/60	67/630	361/120	608320	45979487/435456000
5	1225	1/30	1/60	67/630	361/120	1355904	28468099/120960000
5	1525	1/30	1/60	67/630	361/120	3628740	1462805573/2322432000
5	1600	1/30	1/60	67/630	361/120	4505394	18161994113/23224320000
5	2725	1/30	1/60	67/630	361/120	49421124	11068043761/129024000
5	3025	1/30	1/60	67/630	361/120	872059200/11	4394269973/319334400

Case (d)

5.13. We shall finally consider triality forms ${}^{3,6}D_4$. Let now G be a triality form over a totally real number field k such that for any real place v of k either G is anisotropic at v (or, equivalently, $G(k_v)$ is compact, and over k_v , G is an inner form of a split group), or the symmetric space of $G(k_v)$ is hermitian, in this case the real rank of G at v is 2, and it is again an inner form of a split group over k_v . We conclude from this that there is a totally real cubic extension ℓ of k such that over the normal closure of ℓ/k , G is an inner form of a split group. If $k = \mathbb{Q}$, then as at the unique real place of k , the relative rank of G is 2, we conclude that for every place v of $k = \mathbb{Q}$, in the Tits index of G over k_v , the central vertex is distinguished, and then it follows from Proposition 7.1 of [PR] that G is isotropic over $k = \mathbb{Q}$. But as the arithmetic subgroups we are considering are cocompact, by Godement criterion, G is anisotropic over k . *Therefore, we infer that $k \neq \mathbb{Q}$.*

The exponents of G are 1, 3, 3, and 5 (3 has multiplicity 2).

$$\mu(\mathcal{G}/\Lambda) = D_k^{14} (D_\ell/D_k^3)^{7/2} \left(\frac{4320}{(2\pi)^{16}} \right)^d \mathcal{E},$$

where

$$\mathcal{E} = \prod_{v \in V_f} e'(P_v) \cdot \zeta_k(2) \zeta_k(6) L_{\ell|k}(4),$$

with $L_{\ell|k}(4) := \zeta_\ell(4)/\zeta_k(4)$.

From functional equation, the expression for \mathcal{R} is given by

$$\mathcal{R} = 2^{-2d} \zeta_k(-1) \zeta_\ell(-3) L_{\ell|k}(-3).$$

Let \mathcal{T} be the set of nonarchimedean places v of k which are unramified in ℓ and P_v is not a hyperspecial parahoric subgroup of $G(k_v)$. Then

$$[\Gamma : \Lambda] \leq h_{\ell,2} \cdot 2^{2(d+r+\#\mathcal{T})} D_\ell/D_k^3.$$

We note that for all nonarchimedean v , $e'(P_v) \geq 1$, and for $v \in \mathcal{T}$, $e'(P_v) > 4$.

Letting $A = (2\pi)^{16}/4320$ and using the above estimates as before, we conclude that

$$\begin{aligned} D_k^{1/d} &< w_1(d, h_{\ell,2}) := [(4A)^d \cdot h_{\ell,2}]^{1/14d} \\ D_k^{1/d} &< w_3(d, \delta) := [(50\delta(1+\delta)) \left(\frac{A\zeta^3(1+\delta)\Gamma^3(\frac{1+\delta}{2})}{2e^{1.38\pi\frac{3(1+\delta)}{2}}} \right)^d]^{1/(25-3\delta)} \\ D_\ell^{1/3d} &< x_1(d, D_k, h_{\ell,2}) := (h_{\ell,2}(4A)^d D_k^{-\frac{13}{2}})^{\frac{2}{15d}} \\ D_\ell^{1/3d} &< x_3(d, D_k, \delta) := \left[\frac{50\delta(1+\delta)}{D_k^{\frac{13}{2}}} \left(\frac{A\zeta^3(1+\delta)\Gamma^3(\frac{1+\delta}{2})}{2e^{1.38\pi\frac{3}{2}(1+\delta)}} \right)^d \right]^{1/(3d(4-\delta))} \end{aligned}$$

Note that fixed values of δ , all the expressions on the right hand side of the equations are decreasing in d when the value of δ (respectively, $h_{\ell,2}$) is fixed. From direct computation, $D_k^{1/d} \leq w_3(4, 1.642) < 5.03$, which contradicts Proposition 1. Hence there is no solution for $d = 4$ and hence for $d \geq 4$.

Consider now $d = 3$. In such case, as the smallest discriminant of a totally real cubic is 49, the above equations give $D_\ell^{1/2d} < x_3(3, 49, 1.14) < 9.5$, which contradicts the lower bound of degree 9 totally real number field given by Table IV in [Ma]. Hence it does not exist.

Consider now $d = 2$. In such case, as the smallest discriminant of a totally real quadratic field is 5, the above equations give $D_\ell^{1/2d} < x_3(2, 5, 0.74) < 22.2$. According to [Ma], $N_r(21) > 22.3$. Hence from Hilbert Class Field consideration, $h_\ell \leq \lfloor 20/6 \rfloor = 3$ and $h_{\ell,2} \leq 2$. The above equations give $D_\ell^{1/2d} < x_1(2, 5, 2) < 10.4$. According to [Ma], $N_r(8) > 10.5$. Hence from Hilbert Class Field consideration, $h_\ell \leq \lfloor 8/6 \rfloor = 1$. The above equations give $D_\ell^{1/2d} < x_1(2, 5, 1) < 9.896$. It follows that $D_\ell \leq 9.896^6 < 939200$.

Observe also that $x_1(2, D_k, 1)$ is decreasing in D_k . From direct computation, suppose $D_k \geq 8$, $D_\ell^{1/3d} \leq x_1(2, 8, 1) < 8.073$, which is smaller than the lower bound of $N_r(6)$ given by Proposition 1. Hence D_k can only take the values of 5.

For $D_k = 5$, we observe from table t66.001 of [1] that there are 11 totally real sextics with discriminant bounded as above. For ℓ to be an extension of degree three of k , it is necessary that D_ℓ/D_k^2 is an integer. Going through the list of 29 (respectively 4) number fields for $D_k = 5$ (respectively $D_k = 8$), we are left with four possibilities for D_ℓ for $D_k = 5$, with discriminant given by 300125, 485125, 722000 and 820125. Among the four sextics mentioned, only the first one contain $\mathbb{Q}(\sqrt{5})$ as a subfield. Here is the final data, where $k = \mathbb{Q}(\sqrt{5})$.

D_ℓ	ℓ	\mathcal{R}
300125	$x^6 - x^5 - 7x^4 + 2x^3 + 7x^2 - 2x - 1$	5426717059/33075

From the value of \mathcal{R} given by the last entry above and Proposition 1, we conclude that there is no candidate for arithmetic fake compact hermitian symmetric spaces for Case (d) as well.

5.12. In conclusion, there is no arithmetic fake compact hermitian symmetric space of type D_n for $n \geq 4$.

6. G of type 2E_6

6.1. In this section G is of type 2E_6 . Its dimension is 78 and the complex dimension of the symmetric space of $\mathcal{G} = \prod_{j=1}^r G(k_{v_j})$ is $16r$. Let

$$A = \frac{(2\pi)^{42}}{4!5!7!8!11!}.$$

The Brauer-Siegel Theorem for the totally complex number field ℓ asserts that for all real $\delta > 0$,

$$(33) \quad h_\ell R_\ell \leq w_\ell \delta (1 + \delta) \Gamma(1 + \delta)^d ((2\pi)^{-2d} D_\ell)^{(1+\delta)/2} \zeta_\ell(1 + \delta),$$

where R_ℓ is the regulator of ℓ and w_ℓ is the number of roots of unity contained in ℓ . Using this, from bound (20) we obtain

$$(D_k D_\ell)^{13} < h_\ell A^d \leq \frac{\delta(1 + \delta) A^d \Gamma(1 + \delta)^d D_\ell^{(1+\delta)/2} \zeta_\ell(1 + \delta)}{(R_\ell/w_\ell)(2\pi)^{d(1+\delta)}}.$$

Hence,

$$D_k^{13} D_\ell^{13 - \frac{1+\delta}{2}} < \frac{\delta(1 + \delta) A^d \Gamma(1 + \delta)^d \zeta_\ell(1 + \delta)}{(R_\ell/w_\ell)(2\pi)^{d(1+\delta)}}.$$

As $D_k^2 \leq D_\ell$, and $\zeta_\ell(1 + \delta) \leq \zeta(1 + \delta)^{2d}$, we conclude that

$$D_k^{38-\delta} < \frac{\delta(1 + \delta) A^d \Gamma(1 + \delta)^d \zeta(1 + \delta)^{2d}}{(R_\ell/w_\ell)(2\pi)^{d(1+\delta)}}.$$

Therefore,

$$D_k^{1/d} < \left[\left\{ A \frac{\Gamma(1 + \delta) \zeta(1 + \delta)^2}{(2\pi)^{1+\delta}} \right\} \cdot \left\{ \frac{\delta(1 + \delta)}{R_\ell/w_\ell} \right\}^{1/d} \right]^{1/(38-\delta)}.$$

Using the lower bound $R_\ell \geq 0.02 w_\ell e^{0.1d}$ due to R. Zimmert [Z], we obtain from this the following

$$(34) \quad D_k^{1/d} < f(d, \delta) := \left[\left\{ A \frac{\Gamma(1 + \delta) \zeta(1 + \delta)^2}{(2\pi)^{1+\delta} e^{0.1}} \right\} \cdot \{50\delta(1 + \delta)\}^{1/d} \right]^{1/(38-\delta)}.$$

From bound (20) we also obtain,

$$(35) \quad D_\ell/D_k^2 < \left[A^d \frac{h_\ell}{D_k^{39}} \right]^{1/13}.$$

Furthermore, using (25) and Zimmert's bound $R_\ell \geq 0.02 w_\ell e^{0.1d}$, we get from this that

$$(36) \quad D_\ell/D_k^2 < \mathfrak{p}(d, D_k, \delta) := \left[\left\{ A \frac{\Gamma(1 + \delta) \zeta(1 + \delta)^2}{(2\pi)^{1+\delta} e^{0.1}} \right\} \cdot \left\{ \frac{50\delta(1 + \delta)}{D_k^{38-\delta}} \right\}^{1/d} \right]^{2d/(25-\delta)}.$$

6.2. For a fixed δ , $f(d, \delta)$ clearly decreases as d increases. By a direct computation we find that $f(3, 2) < 2.3$, and hence for all $d \geq 3$,

$$D_k^{1/d} < f(d, 2) \leq f(3, 2) < 2.3.$$

But according to Proposition 2, for totally real number fields of degree $d \geq 3$, $D_k^{1/d} > 3.65$, so we conclude that $d \leq 2$.

Assume now that $d = 2$. Then $D_k^{1/2} \leq f(2, 1.94) < 2.4$. Therefore $D_k < 6$ and hence $D_k = 5$. It follows from bound (28) with $\delta = 1.9$ that $D_\ell/D_k^2 < \mathfrak{p}(2, 5, 1.9) < 1.4$. Hence $D_\ell/D_k^2 = 1$ and $D_\ell = 25$, which contradicts the bound given by Proposition 2. Hence, $d = 1$ and $k = \mathbb{Q}$.

It is known, and follows, for example, from Proposition 7.1 of [PR], that a \mathbb{Q} -group G of type 2E_6 , which at the unique real place of \mathbb{Q} is the outer form of rank 2 (this is the form ${}^2E_{6,2}^{16'}$ which gives rise to a hermitian symmetric space), is isotropic over \mathbb{Q} . This contradicts the fact that G is anisotropic over \mathbb{Q} (1.5), and hence we conclude that groups of type 2E_6 do not give rise to arithmetic fake compact hermitian symmetric spaces.

7. G of type E_7

7.1. In this section G is assumed to be of type E_7 . Its dimension is 133, the exponents are 1, 5, 7, 9, 11, 13 and 17. The dimension of the symmetric space X of $\mathcal{G} = \prod_{j=1}^r G(k_{v_j})$ is $27r$.

Let

$$B = \prod_{j=1}^7 \frac{(2\pi)^{m_j+1}}{m_j!}.$$

From (19) we obtain the following:

$$D_k^{1/d} < [2B(h_{k,2}/2)^{1/d}]^{2/133}.$$

Using the Brauer-Siegel Theorem for totally real number fields (see 4.1), and the obvious bound $\zeta_k(1 + \delta) \leq \zeta(1 + \delta)^d$, we obtain

$$(37) \quad D_k^{1/d} < \left[\left\{ B \frac{\Gamma((1 + \delta)/2)\zeta(1 + \delta)}{\pi^{(1 + \delta)/2}} \right\} \cdot \left\{ \frac{\delta(1 + \delta)}{R_k} \right\}^{1/d} \right]^{2/(132 - \delta)}.$$

Now using the lower bound $R_k \geq 0.04 e^{0.46d}$ due to R. Zimmert [Z] again, we get

$$(38) \quad D_k^{1/d} < \phi(d, \delta) := \left[\left\{ B \frac{\Gamma((1 + \delta)/2)\zeta(1 + \delta)}{\pi^{(1 + \delta)/2} e^{0.46d}} \right\} \cdot \{25\delta(1 + \delta)\}^{1/d} \right]^{2/(132 - \delta)}.$$

7.2. For a fixed $\delta \geq 0.04$, $\phi(d, \delta)$ clearly decreases as d increases. By a direct computation we see that $\phi(2, 4) < 2$, and hence for all totally real number field k of degree $d \geq 2$,

$$D_k^{1/d} < \phi(d, 4) \leq \phi(2, 4) < 2.$$

From this bound and Proposition 2 we conclude that d can only be 1, i.e., $k = \mathbb{Q}$. But then $r = 1$ and the complex dimension of the associated symmetric space X

is 27. Then the Euler-Poincaré characteristic of any quotient of X by a cocompact torsion-free discrete subgroup of $\overline{\mathcal{G}}$ is negative (1.3), and hence it cannot be a fake compact hermitian symmetric space. Another way to eliminate this case is to observe that an absolutely simple \mathbb{Q} -group of type E_7 is isotropic if it is isotropic over \mathbb{R} (this result follows, for example, from Proposition 7.1 of [PR]).

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UNIVERSITY OF MICHIGAN, ANN ARBOR, MI 48109
e-mail: gprasad@umich.edu

PURDUE UNIVERSITY, WEST LAFAYETTE, IN 47907
email: yeung@math.purdue.edu