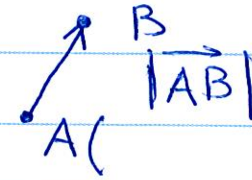
§12.2 Vectors

magnitude : mass, length, time

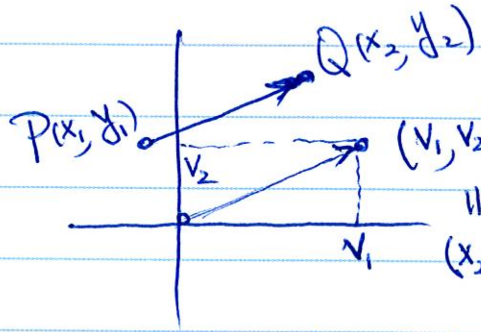
magnitude & direction : force, displacement, ... <u>vector</u>

component form

<u>vector</u> $\Leftrightarrow$ directed line segment $\overrightarrow{AB}$



$\vec{v}_1 = \vec{v}_2 \Leftrightarrow |\vec{v}_1| = |\vec{v}_2|$ and in the same direction



$(v_1, v_2) = \vec{v}$ <u>component form</u>

$\parallel$

$(x_2 - x_1, y_2 - y_1)$

$\overrightarrow{PQ}$ has the same direction and length as $\vec{v}$

$$\Longrightarrow \begin{cases} x_2 = x_1 + v_1 \\ y_2 = y_1 + v_2 \end{cases} \Longrightarrow (v_1, v_2) = \left(x_2 - x_1, y_2 - y_1\right)$$

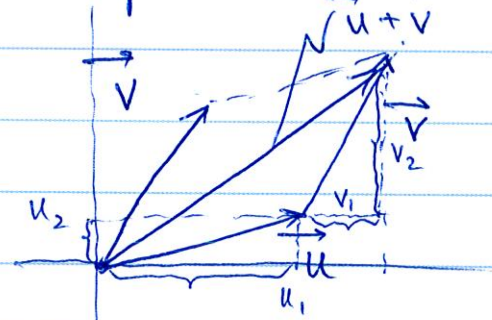
<u>magnitude</u> $|\vec{v}| = \sqrt{v_1^2 + v_2^2}$

Vector Algebra Operations

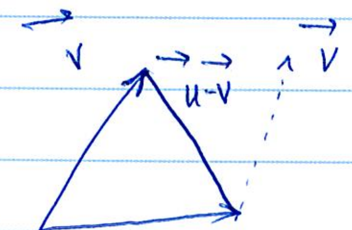
$$\vec{u} = (u_1, u_2, u_3), \quad \vec{v} = (v_1, v_2, v_3)$$

addition $\vec{u} + \vec{v}$

scalar multiplication $k\vec{u}$



difference $\vec{u} - \vec{v} = \vec{u} + (-\vec{v})$



$$\vec{u} = \vec{v} + (\vec{u} - \vec{v})$$

Properties

$\vec{u} + \vec{v} = \vec{v} + \vec{u}$ $\qquad$ $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$

$\vec{u} + \vec{0} = \vec{u}$ $\qquad$ $\vec{u} + (-\vec{u}) = \vec{0}$

$0 \cdot \vec{u} = \vec{0}$ $\qquad$ $1\vec{u} = \vec{u}$

$a(b\vec{u}) = (ab)\vec{u}$ $\qquad$ $a(\vec{u} + \vec{v}) = a\vec{u} + a\vec{v}$

$(a+b)\vec{u} = a\vec{u} + b\vec{u}$

Unit Vector $\vec{i} = (1, 0, 0), \quad \vec{j} = (0, 1, 0), \quad \vec{k} = (0, 0, 1)$

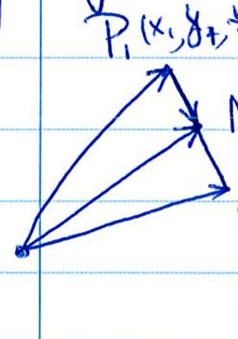
$$\vec{v} = (v_1, v_2, v_3) = v_1\vec{i} + v_2\vec{j} + v_3\vec{k}$$

the direction of $\vec{v} \neq \vec{0}$ $\qquad \dfrac{\vec{v}}{|\vec{v}|}$ unit vector

$\vec{v} \neq \vec{0}$ direction $\dfrac{\vec{v}}{|\vec{v}|}$

length $|\vec{v}|$ $\left(\dfrac{\vec{v}}{|\vec{v}|}\right)$

Midpt of a Line Segment



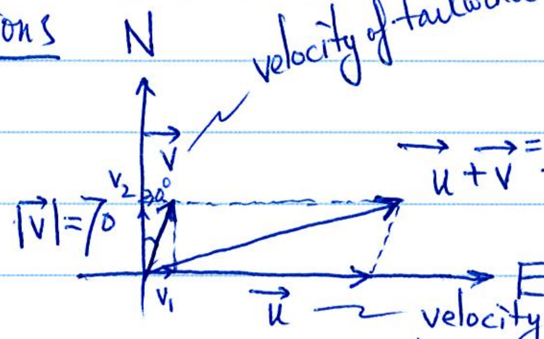
$M\left(\dfrac{x_1+x_2}{2}, \dfrac{y_1+y_2}{2}, \dfrac{z_1+z_2}{2}\right)$

$\frac{1}{2}\left(\overrightarrow{P_1P_2}\right) = \frac{1}{2}\left[\overrightarrow{OP_2} - \overrightarrow{OP_1}\right]$

$\overrightarrow{OM} = \overrightarrow{OP_1} + \overset{\parallel}{\overrightarrow{P_1M}}$

Applications

Ex. 8



$\vec{u} + \vec{v} = ?$ ~ velocity of airplane w.r.t. the ground

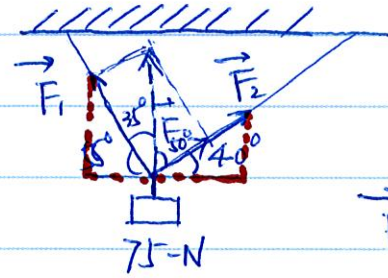
$\vec{u}$ ~ velocity of airplane

$|\vec{u}| = 500$ mph

$\vec{u} = (500, 0)$

Given $\vec{v} = (70\cos 60^\circ, 70\sin 60^\circ) = (35, 35\sqrt{3})$

Ex. 9



Given $\vec{F} = \vec{F_1} + \vec{F_2} = (0, 75)$

Find $\vec{F_1} = ?$, $\vec{F_2} = ?$

$\vec{F_1} = \left(-|\vec{F_1}|\cos 55^\circ, |\vec{F_1}|\sin 55^\circ\right)$, $\vec{F_2} = \left(|\vec{F_2}|\cos 40^\circ, |\vec{F_2}|\cos 40^\circ\right)$

$(0, 75) = \vec{F_1} + \vec{F_2} \Rightarrow |\vec{F_1}| \approx 57.67\,N$, $|\vec{F_2}| \approx 43.18\,N$

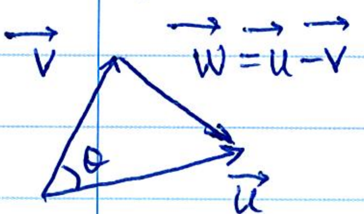
§12.3 The Dot Product

dot product $\vec{u} = (u_1, u_2, u_3)$, $\vec{v} = (v_1, v_2, v_3)$

$$\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + u_3 v_3 \qquad \text{Definition}$$

$$= |\vec{u}|\,|\vec{v}| \cos\theta \qquad \text{Thrm 1}$$

Proof of Thrm 1



Law of Cosine

$$|\vec{w}|^2 = |\vec{u}|^2 + |\vec{v}|^2 - 2|\vec{u}|\,|\vec{v}| \cos\theta$$

$\parallel$ quadratic form

$$|\vec{u} - \vec{v}|^2 = |\vec{u}|^2 + |\vec{v}|^2 - \boxed{\cancel{2|\vec{u}|\,|\vec{v}|}}\; 2\,\vec{u} \cdot \vec{v}$$

Ex. 1 & 2 & 3

Orthogonal Vectors $\vec{u} \perp \vec{v} \iff \vec{u} \cdot \vec{v} = 0$

Ex. 4

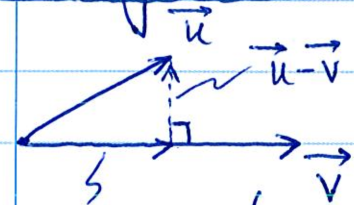
Properties $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$, $(c\vec{u}) \cdot \vec{v} = \vec{u} \cdot (c\vec{v}) = c(\vec{u} \cdot \vec{v})$

$\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$, $\vec{u} \cdot \vec{u} = |\vec{u}|^2$

$\vec{0} \cdot \vec{u} = \vec{0}$

$$|\vec{u} \pm \vec{v}|^2 = |\vec{u}|^2 + |\vec{v}|^2 \pm 2\,\vec{u} \cdot \vec{v}$$

Projection of $\vec{u}$ onto $\vec{v}$



$$\text{Proj}_{\vec{v}}\vec{u} = \left(\vec{u}\cdot\frac{\vec{v}}{|\vec{v}|}\right)\frac{\vec{v}}{|\vec{v}|}$$

$$\text{Proj}_{\vec{v}}\vec{u} = \alpha\vec{v}$$

$$\vec{u}-\vec{v} \perp \alpha\vec{v} = \text{Proj}_{\vec{v}}\vec{u}$$

$$\Downarrow$$

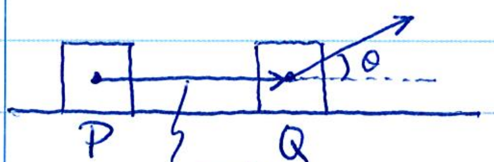
$$0 = (\vec{u}-\vec{v})\cdot\alpha\vec{v}$$

$$\Downarrow$$

$$\alpha = (\vec{u}\cdot\vec{v})\,/\,|\vec{v}|^2$$

Ex. 5 & 6

Work



$\vec{F}$ – const force

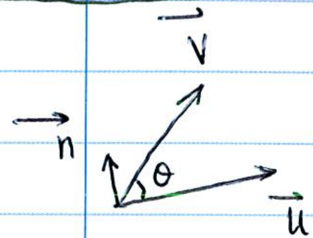
$\vec{D} = \overrightarrow{PQ}$

The work done by a const force $\vec{F}$ acting through a displacement $\vec{D}$

$$W = \vec{F}\cdot\vec{D}$$

Ex. 7

§12.4 The Cross Product



Def $\vec{u} \times \vec{v} = \left(|\vec{u}||\vec{v}|\sin\theta\right)\vec{n}$

$\vec{n}$ — unit vector $\perp$ $\vec{u}$ and $\vec{v}$

Properties

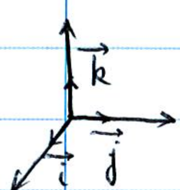
(0) $\vec{u} \,//\, \vec{v} \Rightarrow \vec{u} \times \vec{v} = \vec{0}$

(1) $(r\vec{u}) \times (s\vec{v}) = (rs)(\vec{u} \times \vec{v})$, (2) $\vec{u} \times (\vec{v} + \vec{w}) = \vec{u} \times \vec{v} + \vec{u} \times \vec{w}$

(3) $\vec{v} \times \vec{u} = -(\vec{u} \times \vec{v})$, (4) $(\vec{v} + \vec{w}) \times \vec{u} = \vec{v} \times \vec{u} + \vec{w} \times \vec{u}$

(5) $\vec{0} \times \vec{u} = \vec{0}$, (6) $\vec{u} \times (\vec{v} \times \vec{w}) = (\vec{u} \cdot \vec{w})\vec{v} - (\vec{u} \cdot \vec{v})\vec{w}$

Ex.



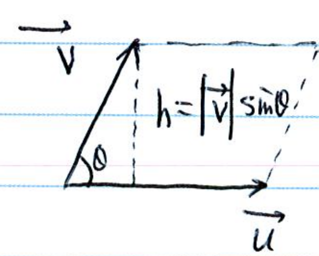
$\vec{i} \times \vec{j} = \vec{k} = -(\vec{j} \times \vec{i})$

$\vec{j} \times \vec{k} = \vec{i} = -(\vec{k} \times \vec{j})$

$\vec{k} \times \vec{i} = \vec{j} = -(\vec{i} \times \vec{k})$

(7) $|\vec{u} \times \vec{v}|$ = area of a parallelogram

$= \left(|\vec{v}|\sin\theta\right)|\vec{u}|$



Determinant Formula for $\vec{u} \times \vec{v}$

$$\vec{u} = u_1 \vec{i} + u_2 \vec{j} + u_3 \vec{k}, \quad \vec{v} = v_1 \vec{i} + v_2 \vec{j} + v_3 \vec{k}$$

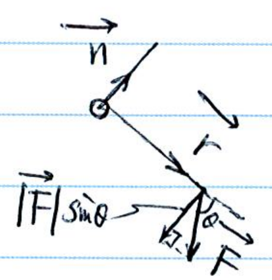
$$\vec{u} \times \vec{v} = \left(u_1 \vec{i} + u_2 \vec{j} + u_3 \vec{k} \right) \times \left(v_1 \vec{i} + v_2 \vec{j} + v_3 \vec{k} \right)$$

$$= (u_2 v_3 - u_3 v_2)\vec{i} - (u_1 v_3 - u_3 v_1)\vec{j} + (u_1 v_2 - u_2 v_1)\vec{k}$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

Ex. 1, 2, 3, 4

Torque



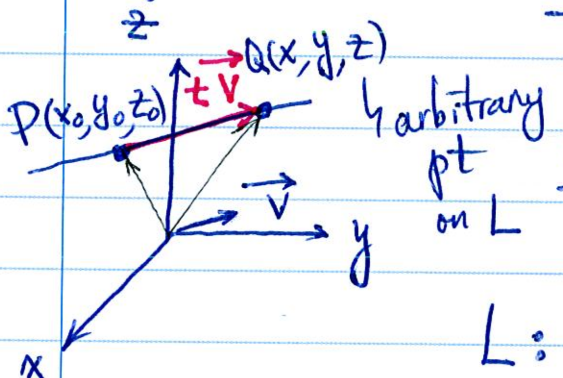
Torque vector $= \left(|\vec{r}|\, |\vec{F}|\, \sin\theta \right) \vec{n} = \vec{r} \times \vec{F}$

Triple Scalar

$$\vec{u} \times \vec{v} \cdot \vec{w} = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

§12.5 Lines and Planes in Space

Equation for a Line



Given (x_0, y_0, z_0) on L

vector $\vec{v} \parallel L$ ~ line

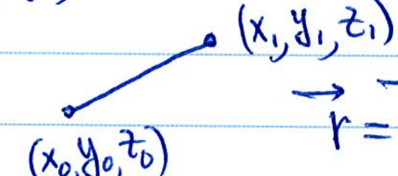
Find equation of L

$$L: \underbrace{(x, y, z)}_{\vec{r}} = \underbrace{(x_0, y_0, z_0)}_{\vec{r}_0} + t\vec{v}$$

$$\begin{cases} x = x_0 + t v_1 \\ y = y_0 + t v_2 \\ z = z_0 + t v_3 \end{cases}$$ parametric equation

- Given (x_0, y_0, z_0), (x_1, y_1, z_1) 2-pts on L $\Longrightarrow \vec{v} = (x_1 - x_0, y_1 - y_0, z_1 - z_0)$

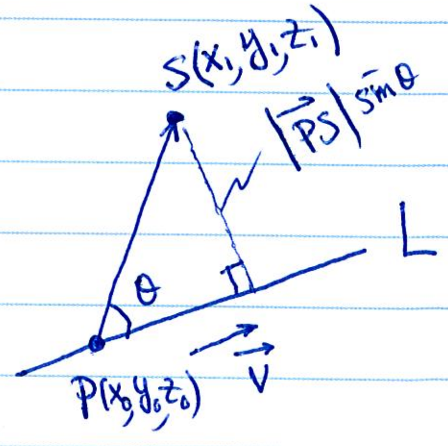
line segment



$\vec{r} = \vec{r}_0 + t\vec{v}$, $t \in [0, 1]$

Ex. 1, 2, 3, 4

Distance from a pt to a line

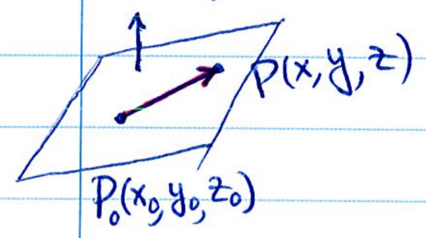


$$d = \frac{|\vec{PS} \times \vec{v}|}{|\vec{v}|} = |\vec{PS}| \sin\theta$$

Ex. 5

Equation for a Plane

$\vec{n} = (A, B, C) \perp$ plane



Given pt $P_0 = (x_0, y_0, z_0)$

vector $\vec{n} \perp$ plane

$$0 = \vec{n} \cdot \overrightarrow{P_0P}$$

$$= A(x - x_0) + B(y - y_0) + C(z - z_0)$$

$$\Leftrightarrow Ax + By + Cz = D$$

Ex. 6, 7

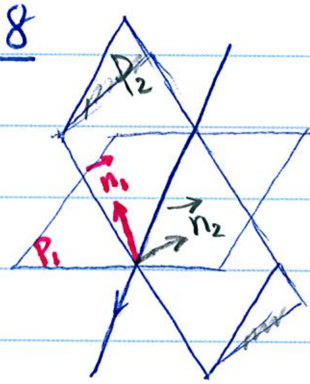
Lines of Intersection

normal vectors of P_1 and P_2

plane P_1 // plane $P_2 \Longleftrightarrow \vec{n}_1 \,//\, \vec{n}_2 \Longleftrightarrow \vec{n}_1 = k\vec{n}_2$

$k \in R$

Ex. 8



Find a vector // the line of intersection of 2 planes

$P_1: 3x - 6y - 2z = 15$

$P_2: 2x + y - 2z = 5$

$$\vec{n}_1 \times \vec{n}_2$$

$$= (14, 2, 15)$$

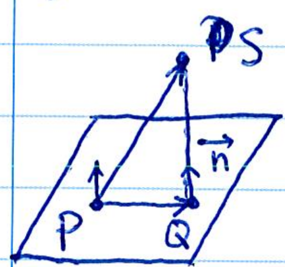
Ex. 9 Find the parametric eq of the line of intersection of 2 planes.

vector $\vec{n}_1 \times \vec{n}_2 = (14, 2, 15)$

pt. $z = 0 \Rightarrow \begin{cases} 3x - 6y = 15 \\ 2x + y = 5 \end{cases} \Rightarrow (3, -1, 0)$

$$\Rightarrow (x, y, z) = (3, -1, 0) + t(14, 2, 15)$$

Distance from a pt to a plane



$$d = \left|\overrightarrow{QS}\right| = \left|\text{Proj}_{\vec{n}}\, \overrightarrow{PS}\right|$$

$$= \left|\left(\overrightarrow{PS} \cdot \frac{\vec{n}}{|\vec{n}|}\right)\frac{\vec{n}}{|\vec{n}|}\right| = \text{abs}\left(\frac{\overrightarrow{PS}\cdot\vec{n}}{|\vec{n}|}\right)$$

$$d = \text{abs}\left(\frac{\overrightarrow{PS}\cdot\vec{n}}{|\vec{n}|}\right)$$

Ex. 11

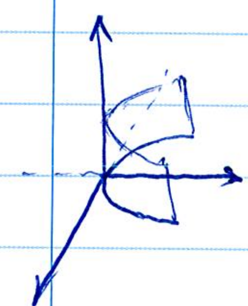
Angles between Planes = angle between their normals.

Ex. 12

§12.6 Cylinders and Quadric Surfaces

Cylinders

2d curve $f(x,y)=c$ defines a 3d cylinder parallel to z-axis.



Quadric Surface

$$Ax^2+By^2+Cz^2+Dx+Ey+Fz=G$$

focus on $Ax^2+By^2+Cz^2+Dz=E$.

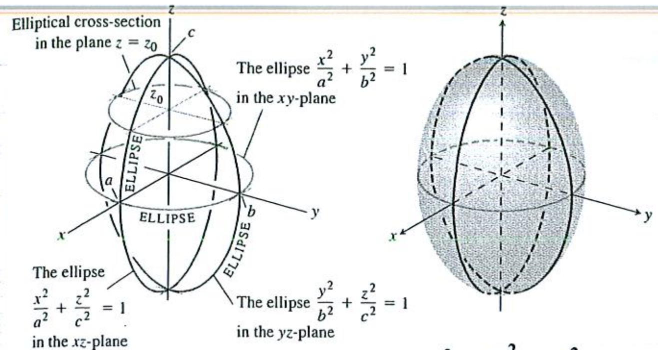
basic quadric surfaces

ellipsoid $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=1$

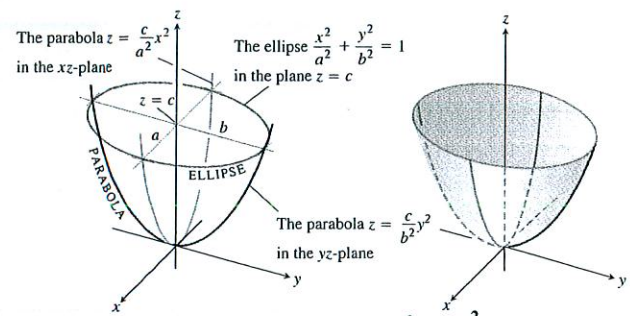
hyperbolic parabloid $\dfrac{y^2}{b^2}-\dfrac{x^2}{a^2}=\dfrac{z}{c}$ with $c>0$

make a transparency of page 717

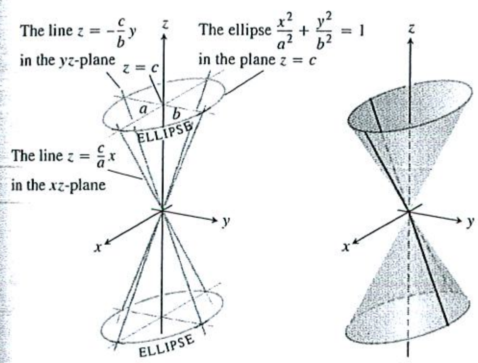
TABLE 12.1 Graphs of Quadric Surfaces



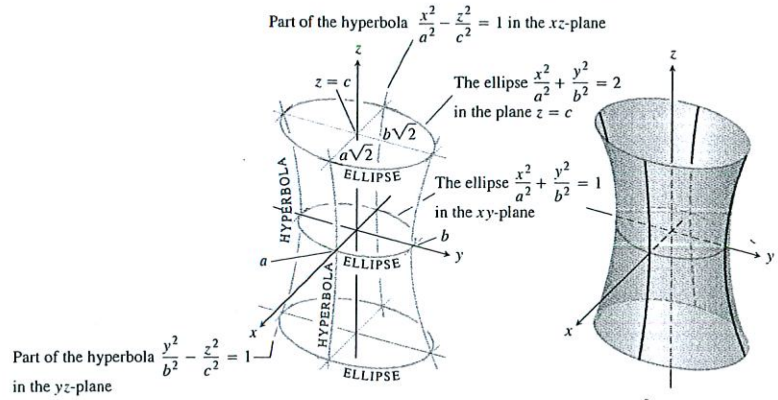
ELLIPSOID $\quad \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$



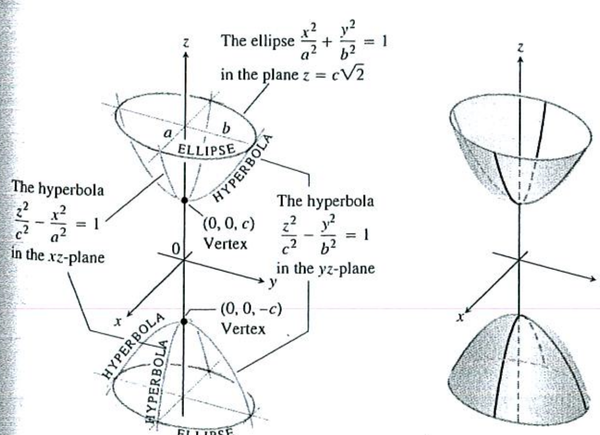
ELLIPTICAL PARABOLOID $\quad \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = \dfrac{z}{c}$



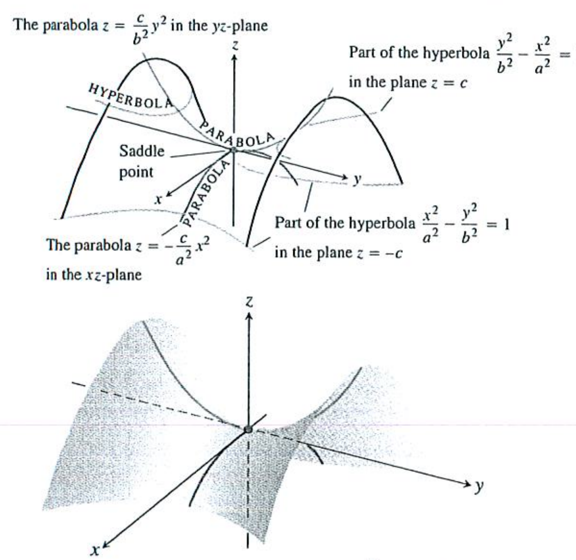
ELLIPTICAL CONE $\quad \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = \dfrac{z^2}{c^2}$



HYPERBOLOID OF ONE SHEET $\quad \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = 1$



HYPERBOLOID OF TWO SHEETS $\quad \dfrac{z^2}{c^2} - \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$



HYPERBOLIC PARABOLOID $\quad \dfrac{y^2}{b^2} - \dfrac{x^2}{a^2} = \dfrac{z}{c}, \quad c > 0$