Overview of Limits

There are two types of limits that we've encountered so far. They are one-sided limits and "overall" limits.

- One-sided limits will **always** exist.
- An "overall" limit will **only** exist when the one-sided limits are **equal**.

In general, you **do not** have to find limits numerically (i.e. plug in x values and see what $f(x)$ approaches; *lessons 2 and 3*) unless a table is given which tells you what x values to plug in to $f(x)$. If a table is given, our normal methods of finding limits probably will not work, and you'll need to use the table.

For this overview, I'm going to assume that **no** table is given.

"Overall" Limit (just called "the limit"):

How to find $\lim_{x \to c} f(x)$

If $f(x)$ is NOT piecewise:

Factor and simplify $f(x)$. Now, find $f(c)$.

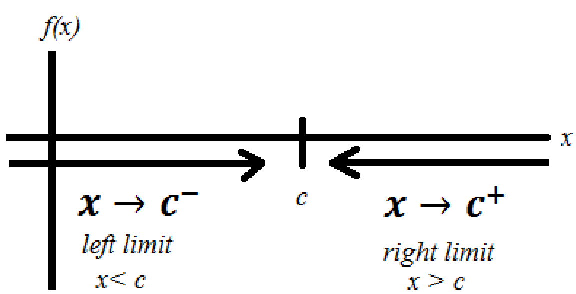
(*Lesson 4: Finding Limits Analytically*)

- **Case 1**: $f(c) = \#$, then $\lim_{x \to c} f(x) = f(c)$
  - Remember that $\frac{0}{\text{nonzero }\#} = 0$
- **Case 2**: $f(c) = \frac{\text{nonzero }\#}{0}$, then $\lim_{x \to c} f(x) = \infty, -\infty, \text{or DNE}$
  - To decide between $\infty, -\infty$, and DNE, look at the one-sided limits.
- **Case 3**: $f(c) = \frac{0}{0}$, then factor, rationalize, or otherwise manipulate $f(x)$ so that $f(c)$ is case 1 or case 2.

If $f(x)$ is piecewise:

Look at the one-sided limits.

One-Sided Limits:



$\lim_{x \to c^-} f(x)$ is the **left limit**. To find this limit, look at $\boldsymbol{x < c}$. From the number line, you can see why we call this the left limit.

$\lim_{x \to c^+} f(x)$ is the **right limit**. To find this limit, look at $\boldsymbol{x > c}$. From the number line, you can see why we call this the right limit.

The + or – in the exponent does NOT make c positive or negative. It only tells you if the values are less than or greater than c.

For piecewise functions, you'll probably be looking at different pieces of the function.

For non-piecewise functions, you'll probably be looking at whether $x - c$ is positive or negative.

Ex. Find $\lim_{x\to 1} \frac{-3}{x-1}$

Here, we have $f(1) = -\frac{3}{1-1} = -\frac{3}{0}$, so we're looking at Case 2 from above. This means the limit will be $\infty, -\infty, \text{or DNE}$, so we have to look at the one-sided limits. The one-sided limits will be $-\infty$ or ∞, so we are mainly concerned with whether the numbers will be positive or negative.

Left Limit: $\lim_{x\to 1^-} \left(\frac{-3}{x-1}\right)$ We are concerned about whether the denominator here will be positive or negative. Since we're choosing values $x < 1$, we have $x - 1 < 0$, so the denominator is negative. Since the numerator is also negative, the numbers will all be positive, so the limit will be positive. As x gets closer to 1, the denominator will be going to 0, so the whole fraction will be getting really big and going towards infinity.

$$\lim_{x\to 1^-} \left(\frac{-3}{x-1}\right) = \left(\frac{-3}{\text{negative number} \to 0}\right) = \infty$$

We can also see the sign to the left of 1 by plugging in $x = 0.9999$, we get $\frac{-3}{0.9999-1} = 30\,000$. We already know the one-sided limits will go to positive or negative infinity. Since the sign is positive, the left limit will go to positive infinity as above.

Right Limit: $\lim_{x\to 1^+} \left(\frac{-3}{x-1}\right)$ Again, we are concerned about whether the denominator will be positive or negative. Since we're choosing values $x > 1$, we have $x - 1 > 0$, so the denominator is positive. Since the numerator is negative, the numbers will all be negative, so the limit will be negative. As x gets closer to 1, the denominator will be going to 0, so the whole fraction will be getting really big and going towards infinity.

$$\lim_{x\to 1^+} \left(\frac{-3}{x-1}\right) = \left(\frac{-3}{\text{positive number} \to 0}\right) = -\infty$$

We can also see the sign to the right of 1 by plugging in $x = 1.0001$, we get $\frac{-3}{1.0001-1} = -30\,000$. We already know the one-sided limits will go to positive or negative infinity. Since the sign is negative, the right limit will go to negative infinity as above.

Now when we look at the overall limit, we have to see if the left and right limits are equal. Since they are not equal, the overall limit is DNE.

Continuity

✳ $f(x)$ is **continuous** at $x = c$ if

① $f(c)$ **is defined**

② $\lim_{x \to c} f(x)$ **exists**

③ $\lim_{x \to c} f(x) = f(c)$

***If $f(x)$ is NOT piecewise*:**

First, find the places where $f(x)$ is undefined. **(Do NOT simplify yet!)** These are almost always found by setting the denominator equal to 0 and solving for x. If $f(x)$ is undefined somewhere, condition ① is not satisfied, so $f(x)$ is **discontinuous** there. The discontinuity will be a hole or a vertical asymptote.

- $f(x)$ has a **hole** if the limit exists and is finite.
- $f(x)$ has a **vertical asymptote** if both one-sided limits are infinite. (The limit may or may not exist.)

The easiest way to determine if you have a hole or a VA is to simplify the function. If you still have $x - c$ in the denominator, $f(x)$ has a VA at $x = c$. If you no longer have $x - c$ in the denominator, $f(x)$ has a hole at $x = c$.

***If $f(x)$ is piecewise*:**

The only places that $f(x)$ will be undefined are usually at the endpoints for the different pieces of the function, so these are the places that you should check for discontinuities. Almost always, discontinuities of a piecewise function will be a hole or a jump.

- $f(x)$ has a **hole** if the limit exists and is finite, but the function is undefined there.
- $f(x)$ has a **jump** if the limit does not exist, and both one-sided limits are finite. (The function may or may not be defined there.)

Ex. Find and classify the discontinuities of $f(x) = \frac{x-1}{x^2-1}$.

First, we need to figure out where $f(x)$ is undefined, so we need to solve $x^2 - 1 = 0$. This gives $(x-1)(x+1) = 0$, so $f(x)$ is undefined at $x = 1$ and $x = -1$. **$\boldsymbol{f(x)}$ is discontinuous at both $\boldsymbol{x = 1}$ and $\boldsymbol{x = -1}$.** Even if we can simplify $f(x)$, it is still undefined at both of those points.

Now, we classify. We can look at the simplified function to classify, so $f(x) = \frac{x-1}{(x-1)(x+1)} = \frac{1}{x+1}$.

Using the shortcut, since the $x - 1$ factors cancel out, $f(x)$ has a hole at $x = 1$, and since the $x + 1$ factor is still in the denominator, $f(x)$ has a vertical asymptote at $x = -1$.

We can also classify by using the definitions. In that case, we need to look at the limits.

<u>x = 1:</u>

$$\lim_{x \to 1} \frac{x-1}{(x-1)(x+1)} = \lim_{x \to 1} \frac{1}{(x+1)} = \left(\frac{1}{2}\right)$$

Then the limit exists and is finite, so there is a hole at $x = 1$.

<u>x = -1:</u>

$$\lim_{x \to -1} \frac{x-1}{(x-1)(x+1)} = \lim_{x \to -1} \frac{1}{(x+1)} = \left(\frac{1}{0}\right)$$

Then we have to look at the one-sided limits to pick between $\infty, -\infty, \text{and DNE}$:

$$\lim_{x \to -1^-} \frac{1}{(x+1)} = \left(\frac{1}{\text{negative } \# \to 0}\right) = -\infty$$

because for $x \to -1^-$ means $x < -1$, so $x + 1 < 0$

$$\lim_{x \to -1^+} \frac{1}{(x+1)} = \left(\frac{1}{\text{positive } \# \to 0}\right) = \infty$$

because for $x \to -1^+$ means $x > -1$, so $x + 1 > 0$

Since both one-sided limits are ∞ or $-\infty$ (i.e. are infinite), there is a vertical asymptote at $x = -1$. (Note that the limit does not exist.)

Ex. Find and classify the discontinuities of $f(x) = \begin{cases} -x, & x < 0 \\ x, & 0 < x \leq 1 \\ x^2 + 1, & x > 1 \end{cases}$

We need to check for discontinuities at $x = 0$ and $x = 1$ because those are the endpoints of the pieces of $f(x)$.

x = 0: First, we need to check the one-sided limits to see if the limit exists.

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} -x = -(0) = 0$$

since $x \to 0^-$ means $x < 0$, we use the top part of $f(x)$.

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} x = 0$$

since $x \to 0^+$ means $x > 0$, we use the middle part of $f(x)$.

$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x)$, so the limit exists and is finite. Now, we need to check if $f(0)$is defined.

When we look at the pieces of $f(x)$, we know when $x < 0$, $f(x) = -x$, and when $x > 0$, $f(x) = x$. Since neither part has an $=$ (or is $\leq$ or $\geq$), $f(0)$ is defined. This means there is a hole at $x = 0$. (If $f(0)$ were defined, then $f(x)$ would be continuous here, i.e., it would NOT have a discontinuity at $x = 0$.)

x = 1: First, we need to check the one-sided limits to see if the limit exists.

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} x = 1 = 1$$

since $x \to 1^-$ means $x < 1$, we use the middle part of $f(x)$.

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (x^2 + 1) = 1^2 + 1 = 2$$

since $x \to 1^+$ means $x > 1$, we use the bottom part of $f(x)$.

$\lim_{x \to 1^-} f(x) \neq \lim_{x \to 1^+} f(x)$, so the limit does not exist, but both one-sided limits are finite. This means there is a jump at $x = 1$.