MA 266 (Fall 2020) **Midterm 1**

Instructor: Otávio Menezes Tuesday, September 29, 2020

Instructions:

- You may use a calculator, but you must show all your work in order to receive credit.
- Be sure to erase or cross out any work that you do not want graded. Circle your final answer.
- When necessary, round answers to two decimal places.

Question:	1	2	3	4	5	6	Total
Points:	15	20	15	20	15	15	100
Score:							

1. Consider the differential equation $y' = y(1-y)$.

   (a) Show that, for any choice of constant C, the function (5)

$$y(t) = \frac{e^t}{e^t + C}$$

   solves the differential equation.

   (b) Is there any solution that is not of the form $y(t) = \frac{e^t}{e^t+C}$? (5)

   (c) Solve the initial value problem (5)

$$y' = y(1-y), \quad y(0) = 3.$$

(a) $\left(\frac{e^t}{e^t+C}\right)' \overset{?}{=} \frac{e^t}{e^t+C}\left(1 - \frac{e^t}{e^t+C}\right)$

$\frac{(e^t+C)e^t - e^t e^t}{(e^t+C)^2} \overset{?}{=} \frac{Ce^t}{(e^t+C)^2}$ this identity is always true

(b) Yes, $y \equiv 0$.

(c) $y(t) = \frac{e^t}{e^t+C}$

$3 = \frac{e^0}{e^0+C} = \frac{1}{1+C} \Rightarrow C = -\frac{2}{3}$

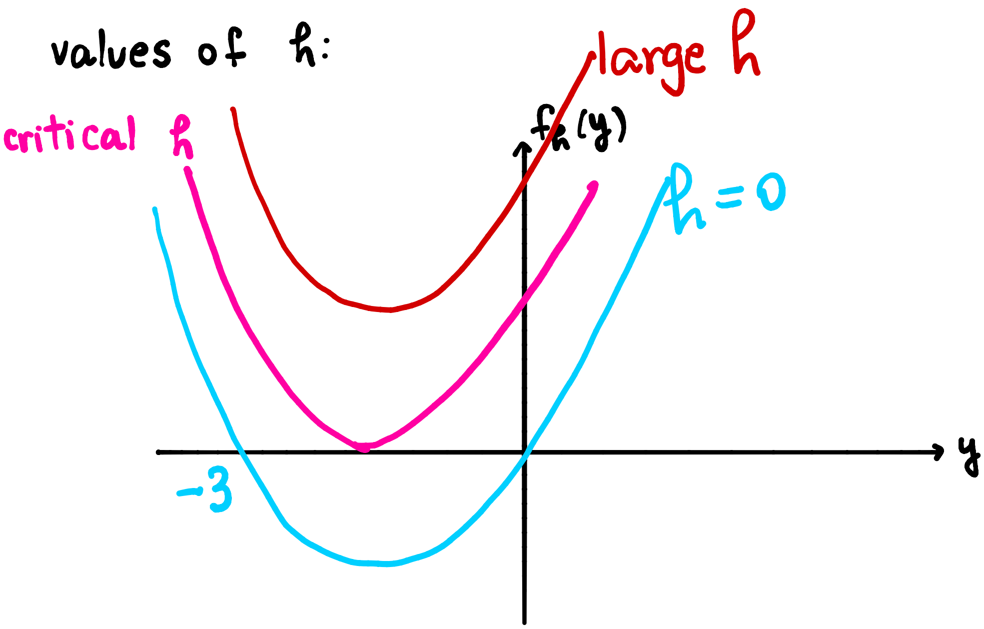
$\Rightarrow \boxed{y(t) = \frac{e^t}{e^t - \frac{2}{3}}}$

2. Consider the following family of initial value problems, indexed by $h > 0$: (20)

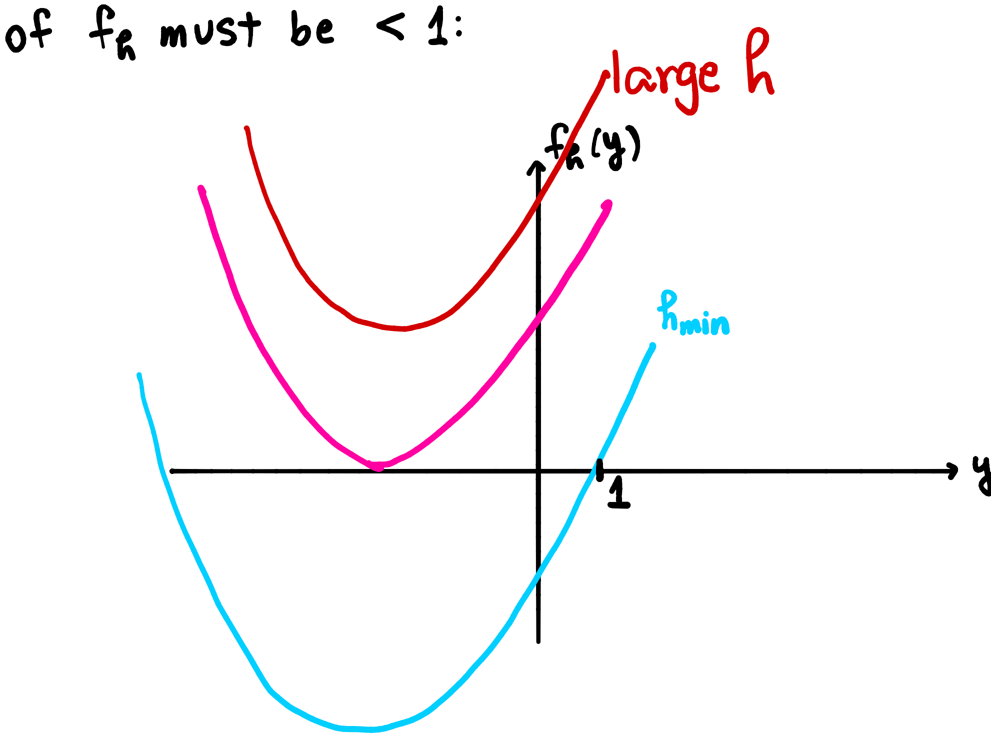
$$\frac{dy}{dt} = y(3+y) + h, \quad y(0) = 1.$$

For which values of h does $\lim_{t\to\infty} y(t) = +\infty$?

Let $f_h(y) := y(3+y) + h$. Here are sketches of f_h for some values of h:



If we want $\lim y(t) = +\infty$, then then the rightmost zero of f_h must be < 1:



What is the rightmost zero?

$0 = y(3+y) + h = y^2 + 3y + h$

$\Rightarrow y = \dfrac{-3+\sqrt{9-4h}}{2}$

Want $1 > \dfrac{-3+\sqrt{9-4h}}{2}$

$\Rightarrow 25 > 9 - 4h$

$\Rightarrow \boxed{h > -4}$.

3. Consider the differential equation

$$(4x^2y^3 + 9x^3y^2)y' + 3xy^4 + 12xy^3 = 0.$$

(a) Show that this equation is not exact. (5)

If it were exact then

$$\frac{\partial}{\partial x}(4x^2y^3+9x^3y^2) = \frac{\partial}{\partial y}(3xy^4+12xy^3)$$

$$8xy^3+27x^2y^2 = 12xy^3+36xy^2 \quad \text{FALSE!}$$

(b) Show that it becomes exact after being multiplied by x. (5)

After multiplying by x, the equation becomes

$$(4x^3y^3+9x^4y^2)dy + (3x^2y^4+12x^2y^3)dx = 0$$

Check for exactness:

$$\frac{\partial}{\partial x}(4x^3y^3+9x^4y^2) \stackrel{?}{=} \frac{\partial}{\partial y}(3x^2y^4+12x^3y^3)$$

$$12x^2y^3+36x^3y^2 \stackrel{?}{=} 12x^2y^3+36x^3y^2 \quad \text{TRUE!}$$

(c) Find the general solution. (5)

We look for an implicit solution $\varphi(x,y)=C$, where

$\frac{\partial\varphi}{\partial x} = 3x^2y^4+12x^3y^3$ and $\frac{\partial\varphi}{\partial y} = 4x^3y^3+9x^4y^2$. $\circledast$

Integrating the first eq. with respect to x,

$\varphi(x,y) = x^3y^4+3x^4y^3+C(y)$ for some function $C(y)$.

Using the second eq. in $\circledast$,

$4x^3y^3+9x^4y^2+C'(y) = 4x^3y^3+9x^4y^2 \Rightarrow C'(y)=0.$

Which gives the implicit solution $\boxed{x^3y^4+3x^4y^3=C}$.

4. A 25-liter tank is filled with water and 3 kg of salt. A salty solution with concentration 5 kg of salt per liter is added to the tank at a rate of 2 liters per minute. Through a separate spout, water is allowed to exit the tank at a rate of 2 liters per minute so that it does not overflow. We may assume the water in the tank is always well-mixed. How much salt is in the tank after 5 minutes? (Round your answer to two decimal places.) (20)

Let $y(t)$ be the amount of salt, in kg, present in the tank after t minutes. We are looking for $y(5)$.

$$y'(t) = 5\times 2 - 2\times\frac{y(t)}{25} = 10 - \frac{2}{25}y(t), \quad y(0) = 3$$

1st order linear equation, can multiply by the integrating factor $\exp\left(\int \frac{2}{25}dt\right) = \exp\left(\frac{2}{25}t\right)$.

$$\frac{d}{dt}\left(y(t)\exp\left(\tfrac{2}{25}t\right)\right) = 10\cdot\exp\left(\tfrac{2}{25}t\right)$$

$$y(t) = 125 + C\cdot\exp\left(-\tfrac{2}{25}t\right)$$

Use $y(0)=3$ to solve for C:

$$3 = 125 + C \Rightarrow C = -122$$

Compute $y(5)$:

$$y(5) = 125 - 122\cdot\exp(-0.4) \approx 43.22$$

<u>ANSWER</u>: after 5 minutes, there will be approximately 43.22 kg of salt in the tank.

5. Consider the differential equation

$$x^2y'' + 2xy' - 12y = 0$$

.

$x = e^v$

(a) Show that the substitution $v = \ln x$ transforms the equation into a linear equation with constant coefficients. (8)

Let $z(v) := y(x)$. Then

$$z'(v) = y'(x)\,x'(v) = y'(x)\,e^v = y'(x)\,x \quad \text{(I)}$$

$$z''(v) = \frac{d}{dv}[y'(x)]x + y'(x)\,x'(v)$$

$$= x\,[y''(x)\,x'(v)] + x\,y'(x) \quad x'(v) = x$$

$$= x^2y''(x) + xy'(x) \quad \text{(II)}$$

From (II), $x^2y''(x) = z''(v) - xy'(x) \overset{\text{(I)}}{=} z''(v) - z'(v)$

From (I), $xy'(x) = z'(v)$

Plug into the equation:

$$(z''(v) - z'(v)) + 2z'(v) - 12z(v) = 0$$

$$z'' + z' - 12z = 0 \qquad \text{constant coefficients}$$

(b) Find the general solution. (7)

The characteristic equation for $z'' + z' - 12z = 0$ is $r^2 + r - 12 = 0$, which factors as $(r+4)(r-3) = 0$. Thus the general solution is

$$z(v) = Ae^{-4v} + Be^{3v}$$

$$\Rightarrow \boxed{x(y) = Ax^{-4} + Bx^3}$$

6. Solve the initial value problem (15)

$$9y'' + 6y' + 4y = 0, \quad y(0) = 3, y'(0) = 4.$$

The characteristic equation is $9r^2+6r+4=0$

Roots $r = \dfrac{-6 \pm \sqrt{36 - 4\times 36}}{18} = \dfrac{-6 \pm 6\sqrt{3}\,i}{18} = -\frac{1}{3} \pm \frac{i}{\sqrt{3}}$

General solution $y(x) = \left(A\cos\left(\frac{x}{\sqrt{3}}\right) + B\sin\left(\frac{x}{\sqrt{3}}\right)\right)e^{-x/3}$

$y'(x) = \left(-\frac{A}{\sqrt{3}}\sin\left(\frac{x}{\sqrt{3}}\right) + \frac{B}{\sqrt{3}}\cos\left(\frac{x}{\sqrt{3}}\right)\right)e^{-x/3} - \frac{1}{3}y(x)$

Use $y(0)=3$, $y'(0)=4$ to solve for A and B.

$y(0) = 3 \Rightarrow A = 3$

$y'(0) = 4 \Rightarrow \frac{B}{\sqrt{3}} - \frac{1}{3}\times 3 = 4 \Rightarrow B = 5\sqrt{3}$

ANSWER: $y(x) = 3\cos\left(\frac{x}{\sqrt{3}}\right) + 5\sqrt{3}\sin\left(\frac{x}{\sqrt{3}}\right)$.