1.2 INTEGRALS AS GENERAL AND PARTICULAR SOLUTIONS

In the 1st lecture, we've seen:

- ↪ what a diff eq is and some examples
- ↪ what a solution and a general solution are

Today we'll do some problems from Section 1.2, learn how to visualize diff eqs and examine assumptions for uniqueness of solutions.

A general solution is a one-parameter family of solutions.

EXAMPLE $\frac{dy}{dx} = 2x+3$, $y(1) = 2$

$$y(x) - y(1) = \int_1^x y'(t)\,dt$$

$$= \int_1^x 2t+3\,dt$$

$$= t^2+3t \Big|_1^{t=x}$$

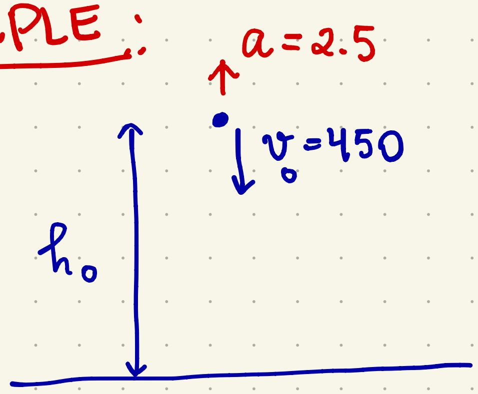
$$= x^2+3x-4$$

$$\Rightarrow y(x) = x^2+3x-4+y(1)$$

$$= x^2+3x-4+2$$

$$= x^2+3x-2$$

EXAMPLE:



v_0: initial speed, in m/s

a: acceleration in m/s^2

(Moon + retrorockets)

h_0: initial height (UNKNOWN)

GIVEN: the ground is reached at speed 0

FIND h_0

Let $h(t)$ be the height at time t seconds.

We know

(I) $h'(t) = -450 + 2.5t$ (at each second you get speed 2.5 m/s)

(II) If $h(t)=0$ then $h'(t)=0$.

Want $h(0)$.

Plan: 1) find t_e such that $h'(t_e)=0$ using (I)

2) plug this t into $h(t_e) - h(0) = \int_0^{t_e} h'(s)\,ds$ and solve for $h(0)$

Carrying out the plan.

1) Solve for t_ℓ in

$$0 = -450 + 2.5\, t_\ell \implies t_\ell = 180 \qquad \left(\text{trick} : \frac{450}{2.5} = \frac{450}{5} \times 2 = 90 \times 2\right)$$

2) $$\underbrace{h(180)}_{\substack{=0 \text{ by} \\ \text{assumption (II)}}} - h(0) = \int_0^{180} -450 + 2.5\, s \, ds = -450 \cdot 180 + 2.5 \cdot \frac{180^2}{2}$$

$$\implies h(0) = 450 \times 180 - 2.5 \times \frac{180^2}{2} = 40{,}500$$

<u>ANSWER</u>: the initial height has to be $40{,}500\,m$.

EXAMPLE: On planet G, a ball dropped from a height of 20ft hits the ground in 2s.

A person can throw a ball straight upward on Earth to a maximum height of 144ft.

How high could this person throw the ball on planet G? Disregard air friction. Take $g = 32\,ft/s^2$ on Earth.

Plan: 1) what is the grav. accel. on planet G?

2) at which speed is the ball thrown upward?

3) how high does it go?

1) Let $x(t)$ be the height of the ball dropped on planet G in ft

let a be the grav. accel. on planet G in ft/s^2

Then $\left.\begin{array}{l} x(0) = 20 \\ x''(t) = -a \end{array}\right\}$ $x(t) - x(0) = \int_0^t x'(s)\,ds = \int_0^t \int_0^s x''(r)\,dr\,ds = \int_0^t \int_0^s -a\,dr\,ds$

$$= \int_0^t -as\,ds = -\frac{at^2}{2}$$

$$\Rightarrow x(t) = 20 - \frac{at^2}{2}$$

When does the ball hit the ground? When $x(t) = 0 \Leftrightarrow \frac{at^2}{2} = 20$

But we know it hits the ground in 2s, so $\frac{a \cdot 2^2}{2} = 20 \Rightarrow \boxed{a = 10}$

2) Let $v(t)$ be the speed of the ball thrown upward on Earth that reaches 144ft!

WANT $v(0)$.

KNOW : when $v(t)=0$, the height is 144ft.

When is $v(t)=0$?

$$v(t) = v(0) - 32t \implies v(t)=0 \text{ when } t = \frac{v(0)}{32}.$$

What is the height at this time?

$$144 = h(t) = \int_0^t h'(s)\,ds = \int_0^t v(s)\,ds = \int_0^{v(0)/32} v(0) - 32s\,ds = \frac{v(0)^2}{32} - 32\,\frac{v(0)^2}{2\cdot 32^2}$$

$$= \frac{v(0)^2}{64}$$

$$\implies v(0) = 8\times 12 = 96$$

3) How high does the ball go?

Now $v(0) = 96$, $v'(t) = -10$, $h(0) = 0$.

When is $v(t)=0$? $v(t) = 96 - 10t \implies v(t)=0$ at time $t = 9.6$

What is the height at $t=9.6$? $h(9.6) - h(0) = \int_0^{9.6} v(s)\,ds = \int_0^{9.6} 96 - 10s\,ds = 96\times 9.6 - 10\times \frac{9.6^2}{2}$

$= 460.8$

✓

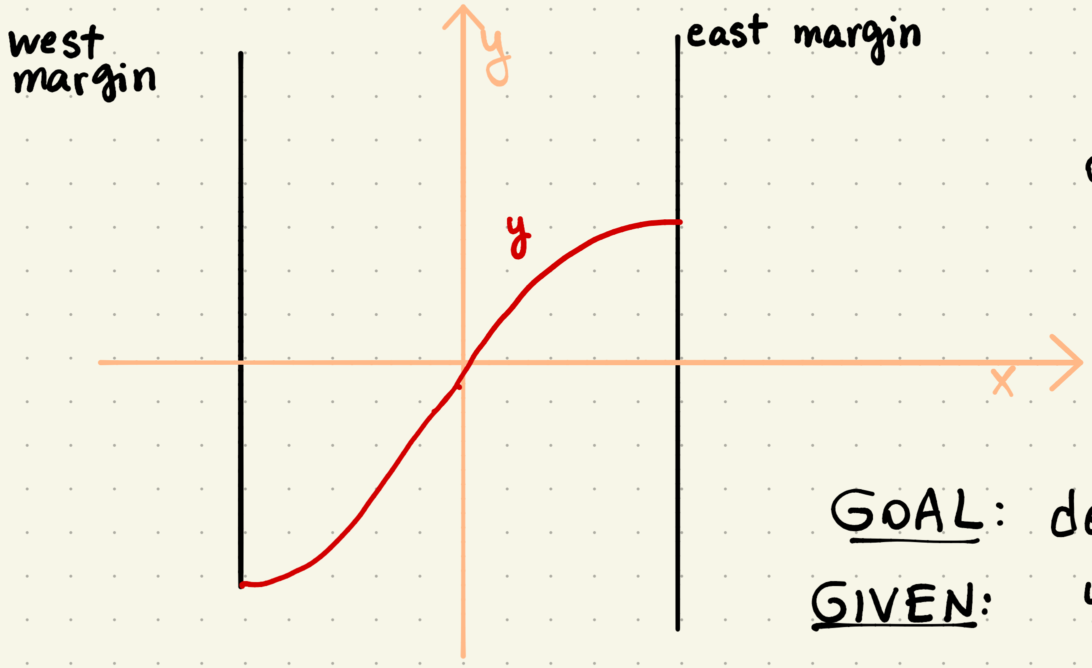
REALITY CHECK: does the answer make sense?

Planet G has weaker gravity, so it makes sense that the ball goes higher.

EXAMPLE: a river of 1mi width flows northward. At distance x from the center, it flows at speed (in mi/h) $v(x) = 9\left(1 - \frac{x^2}{0.5^2}\right)$.

A swimmer crosses the river at constant speed v_s, swimming due east.

How fast does he need to swim in order to drift only 1mile downstream?



Choose a coordinate system whose y axis goes northwards in the middle of the river.

Denote by $y(x)$ the path followed by the swimmer, by v_s the speed of the swimmer, in mi/h

<u>GOAL</u>: determine v_s.

<u>GIVEN</u>: $y(0.5) - y(-0.5) = 1$ (swimmer drifts 1mi downstream)

$$y'(x) = \frac{v(x)}{v_s} \Rightarrow v_s\, y'(x) = 9\left(1 - \frac{x^2}{0.25}\right)$$

Thus, we have the problem

$$v_s\, y'(x) = 9 - 36x^2 \qquad (\mathrm{I})$$
$$y(\tfrac{1}{2}) - y(\tfrac{-1}{2}) = 1 \qquad (\mathrm{II})$$

Integrating (I),

$$v_s\, y(x) = 9x - 12x^3 + C \qquad \text{for some } C.$$

To find C and v_s, use eq (II)

$$y(\tfrac{1}{2}) - y(-\tfrac{1}{2}) = 1 \implies \frac{1}{v_s}\left[\left(\frac{9}{2} - \frac{12}{8} + c\right) - \left(\frac{9}{2} + \frac{12}{8} + c\right)\right] = 1 \implies \boxed{v_s = 6}$$

ANSWER: to drift 1 mi downstream, the swimmer has to go at 6 mi/h.