1.3 SLOPE FIELDS AND SOLUTION CURVES

PREVIOUSLY: → a diff eq. is a possible relationship between functions

$$y' = ay$$
$$f'' = -f$$
$$g'' = \sqrt{1+(g')^2}$$

→ Newton's 2nd law is a diff. eq.

If $x(t)$ is the position of a point mass at time t

the mass suffers a force $F(x)$ at position x

Then $mx''(t) = F(x(t))$.

→ a diff. eq. can have many solutions / several different trajectories are possible under the action of the same forces

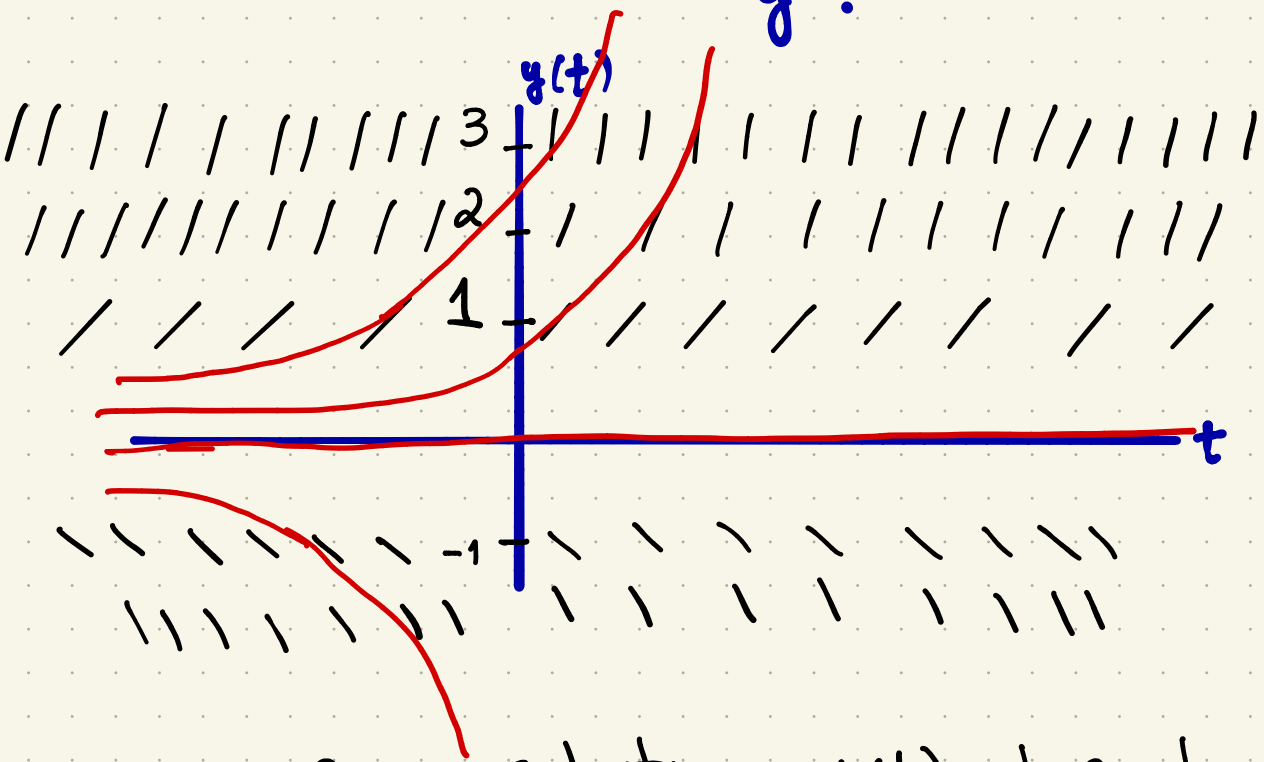
$y' = ay$ has GENERAL SOLUTION $y(t) = Ce^{at}$, C arbitrary parameter

TODAY: → how to visualize the solutions of diff. eqs.

→ criterion for existence and uniqueness of solutions

SLOPE FIELDS:

Look at $y' = y$. What do we know about the GRAPH of y?



"mini tangents"

$y' = y$ so if $y(t) = 1$ then $y'(t) = 1$
(slope of the graph $= 1$)

any solution $y(t)$ has to have the slopes given in the picture

MORE EXAMPLES (ON dfield or Wolfram)

$y' = \sqrt{1+y^2}$

$y' = x + y$

$y' = y - x$

$y' = 2t$

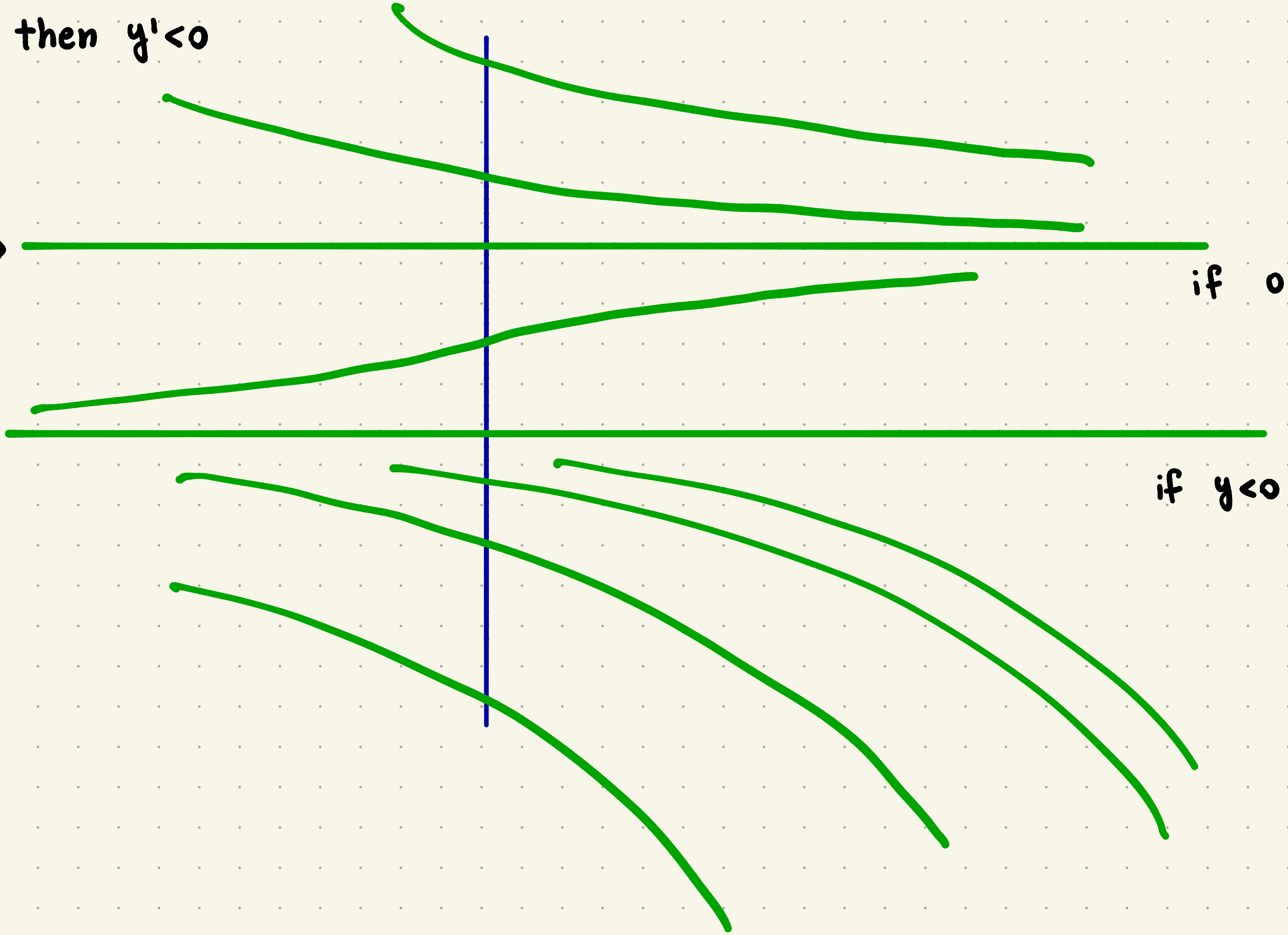
REMARKS: → there is no slope field when the eq. has $y'', y''', \dots$
(we'll learn how to visualise those later, in chapter 3)

→ we have 3 ways of looking at the same problem:

1) solve a diff. eq. $y'(t) = f(t, y(t))$
2) find a graph that matches the mini tangents in the slope field
3) find the TRAJECTORY followed by a point, knowing its velocity

→ in all of the pictures, different solutions do not cross each other, intead they nicely foliate the plane

EXAMPLE: $y' = y(1-y)$



1.3B EXISTENCE AND UNIQUENESS OF SOLUTIONS

ASSERTION: if you know where all the parts of a physical system (think a machine, or the planets, or the atmosphere, or the balls in a billiard game) are, how fast they are moving and what forces they suffer, then you can theoretically predict where they will be at any moment in the future.

As far as this can be formulated in terms of diff. eqs., NOT ALWAYS TRUE, but MOSTLY TRUE.

- EXAMPLE: a diff. eq. whose solutions are <u>not</u> defined for all times

$$\begin{cases} y'(t) = y(t)^2 \\ y(0) = 1 \end{cases}$$

If $y(t)$ is the number of individuals in a population (measured in 1000s, say) and reproduction occurs whenever two individuals meet, then $y'(t) = y(t)^2$ can describe the growth of the population over time

(draw slope field)

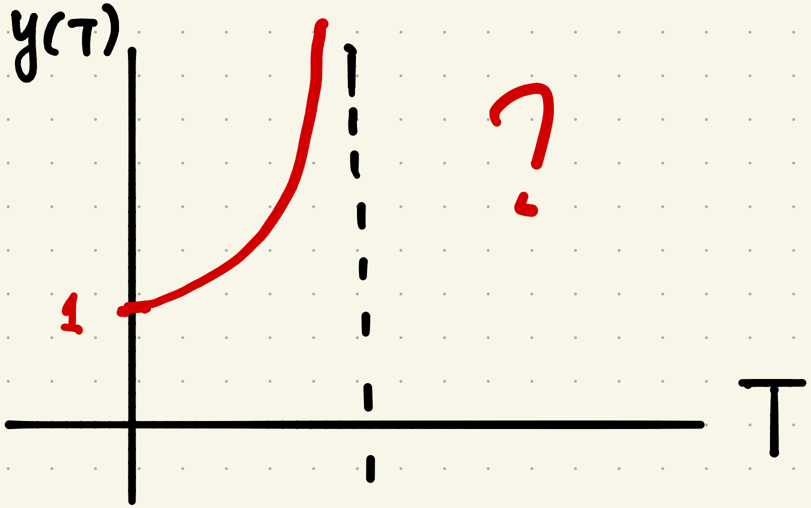
$$\begin{cases} y'(t) = y(t)^2 \\ y(0) = 1 \end{cases}$$ what is $y(5)$?

This is the 1st eq. we'll solve:

$$\frac{y'(t)}{y(t)^2} = 1 \implies -\left(\frac{1}{y(t)}\right)' = 1 \implies \int_0^T \left(\frac{1}{y(t)}\right)' dt = -\int_0^T 1 \, dt$$

$$\implies \frac{1}{y(T)} - \frac{1}{y(0)} = -T$$

$$\implies y(T) = \frac{1}{1-T}$$



Solution $y(T)$ is <u>not</u> defined for $T \geq 1$.

••) A diff. eq. with INFINITELY MANY solutions, even when initial condition is given.

the value of the unknown function and its derivative at a particular point are known

$$\begin{cases} x'(t) = 2\sqrt{x(t)} \\ x(0) = 0 \end{cases}$$ (draw slope field)

Can solve as in the previous example:

$$\frac{x'(t)}{2\sqrt{x(t)}} = 1 \Rightarrow \frac{d}{dt}\sqrt{x(t)} = 1 \xRightarrow{\int_0^T} \sqrt{x(T)} - \sqrt{x(0)} = T$$

$$\Rightarrow x(T) = T^2$$

However, there are more solutions!

$x(T) = 0$ for all T also works!

$$x(T) = \begin{cases} 0, & T < a \\ (T-a)^2, & T \geq a \end{cases}$$ also works!

The particle can "choose" when to leave 0 and start going with the flow.

So when are solutions unique?

PICARD'S THEOREM: Consider the IVP

$$\frac{dy}{dx} = f(x,y), \quad y(a) = b \quad \star$$

IF there is a rectangle R in the xy plane that contains (a,b), f is continuous on R and $\frac{\partial f}{\partial y}$ is continuous on R.

THEN the problem $\star$ has one and only one solution, defined for x in some open interval that contains a.

In the example

$$y' = 2\sqrt{y}, \quad y(0) = 0$$

we have $f(x,y) = 2\sqrt{y}$ and $\frac{\partial f}{\partial y}(x,y) = \frac{1}{2\sqrt{y}}$, not defined at $y=0$.

Example: Consider the IVP

$$x\,y'(x) = y(x), \quad y(a) = b.$$

When does it have a unique solution?

Picard's Thm starts with a diff. eq. in the form $y'(x) = f(x,y)$.

$$x\,y'(x) = y(x)$$

$$y' = \frac{y}{x} =: f(x,y) \quad \text{IF } x \neq 0.$$

Apply Picard's Thm: f is continuous when $x \neq 0$, $\frac{\partial f}{\partial y}(x,y) = \frac{1}{x}$ cont when $x \neq 0$.

$\Longrightarrow$ UNIQUE SOLUTION FOR $a \neq 0$.

What about $a = 0$?

$$0 \cdot y'(0) = y(0) \Rightarrow y(0) = 0$$

Check that $y(x) = 0$ is a solution. Are there more? YES! Any line through $(0,0)$ is a solution: $y(x) = Cx$.

$$x \cdot y'(x) = x \cdot C = y(x).$$