1.3B EXISTENCE AND UNIQUENESS OF SOLUTIONS

ASSERTION: if you know where all the parts of a physical system (think a machine, or the planets, or the atmosphere, or the balls in a billiard game) are, how fast they are moving and what forces they suffer, then you can theoretically predict where they will be at any moment in the future.

As far as this can be formulated in terms of diff. eqs., NOT ALWAYS TRUE, but MOSTLY TRUE.

- EXAMPLE: a diff. eq. whose solutions are <u>not</u> defined for all times

$$\begin{cases} y'(t) = y(t)^2 \\ y(0) = 1 \end{cases}$$

If $y(t)$ is the number of individuals in a population (measured in 1000s, say) and reproduction occurs whenever two individuals meet, then $y'(t) = y(t)^2$ can describe the growth of the population over time

(draw slope field)

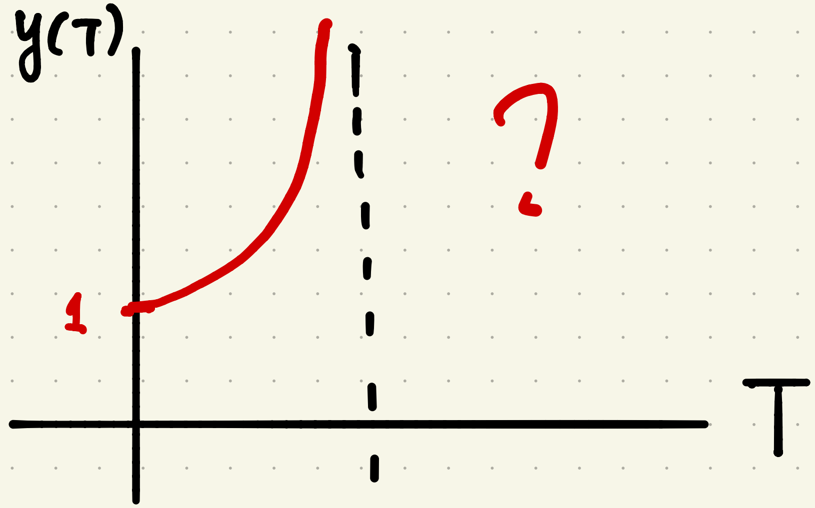
$$\begin{cases} y'(t) = y(t)^2 \\ y(0) = 1 \end{cases}$$ what is $y(5)$?

This is the 1st eq. we'll solve:

$$\frac{y'(t)}{y(t)^2} = 1 \Rightarrow -\left(\frac{1}{y(t)}\right)' = 1 \Rightarrow \int_0^T \left(\frac{1}{y(t)}\right)' dt = -\int_0^T 1 \, dt$$

$$\Rightarrow \frac{1}{y(T)} - \frac{1}{y(0)} = -T$$

$$\Rightarrow y(T) = \frac{1}{1-T}$$



Solution $y(T)$ is not defined for $T \geq 1$.

··) A diff. eq. with INFINITELY MANY solutions, even when initial condition is given.

the value of the unknown function and its derivative at a particular point are known

$$\begin{cases} x'(t) = 2\sqrt{x(t)} \\ x(0) = 0 \end{cases}$$ (draw slope field)

Can solve as in the previous example:

$$\frac{x'(t)}{2\sqrt{x(t)}} = 1 \Rightarrow \frac{d}{dt}\sqrt{x(t)} = 1 \overset{\int_0^T}{\Longrightarrow} \sqrt{x(T)} - \sqrt{x(0)} = T$$

$$\Rightarrow x(T) = T^2$$

However, there are more solutions!

$x(T) = 0$ for all T also works!

$$x(T) = \begin{cases} 0, & T < a \\ (T-a)^2, & T \geq a \end{cases}$$ also works!

The particle can "choose" when to leave 0 and start going with the flow.

So when are solutions unique?

PICARD'S THEOREM: Consider the IVP

$$\frac{dy}{dx} = f(x,y), \quad y(a) = b \qquad \star$$

IF there is a rectangle R in the xy plane that contains (a,b), f is continuous on R and $\frac{\partial f}{\partial y}$ is continuous on R.

THEN the problem $\star$ has one and only one solution, defined for x in some open interval that contains a.

In the example

$$y' = 2\sqrt{y}, \quad y(0) = 0$$

we have $f(x,y) = 2\sqrt{y}$ and $\frac{\partial f}{\partial y}(x,y) = \frac{1}{2\sqrt{y}}$, not defined at $y=0$.

Example: Consider the IVP

$$x\,y'(x) = y(x), \quad y(a) = b.$$

When does it have a unique solution?

Picard's Thm starts with a diff. eq. in the form $y'(x) = f(x,y)$.

$$x\,y'(x) = y(x)$$

$$y' = \frac{y}{x} =: f(x,y) \quad \text{IF } x \neq 0.$$

Apply Picard's Thm: f is continuous when $x \neq 0$, $\frac{\partial f}{\partial y}(x,y) = \frac{1}{x}$ cont when $x \neq 0$.

$\Longrightarrow$ UNIQUE SOLUTION FOR $a \neq 0$.

What about $a = 0$?

$$0 \cdot y'(0) = y(0) \Rightarrow y(0) = 0$$

Check that $y(x) = 0$ is a solution. Are there more? YES! Any line through $(0,0)$ is a solution: $y(x) = Cx$.

$$x \cdot y'(x) = x \cdot C = y(x).$$

EXAMPLE: consider the IVP

$$y' = \sqrt{1-y^2}, \quad y(a) = b. \quad \star$$

For which values of a and b is there a unique solution?

Can use Picard's Thm.

$y' = \sqrt{1-y^2} =: f(x,y)$ (does not depend on x)

f is continuous everywhere

$\frac{\partial f}{\partial y}(x,y) = \frac{\partial}{\partial y}\sqrt{1-y^2} = \frac{2y}{2\sqrt{1-y^2}}$ undefined at $y=1$ and $y=-1$, continuous elsewhere

Picard $\Rightarrow$ uniqueness of solution for $\star$ if $b \neq 1$, $b \neq -1$.

Are there solutions if $y(a) = 1$?

$y'(1) = \sqrt{1-y(1)^2} = 0 \quad \Rightarrow$ A solution is $y \equiv 1$ (constant function)

Are there more solutions? Look at the slope field.

$$y' = \sqrt{1-y^2} \Longrightarrow \frac{dy}{\sqrt{1-y^2}} = 1 \Rightarrow \int \frac{dy}{\sqrt{1-y^2}} = t + C \Longrightarrow \sin^{-1} y(t) = t + C \Rightarrow y(t) = \sin(t+C)$$

$y = \sin\theta$
$dy = \cos\theta \, d\theta$

However, the change of variables $y = \sin\theta$ only works for $\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$, so our solution is only defined for $-\frac{\pi}{2} \leq t + C \leq \frac{\pi}{2}$. Can extend to other values of t by letting $y(t) = 1$ if $t + C > \frac{\pi}{2}$ and $y(t) = -1$ if $t + C < -\frac{\pi}{2}$.

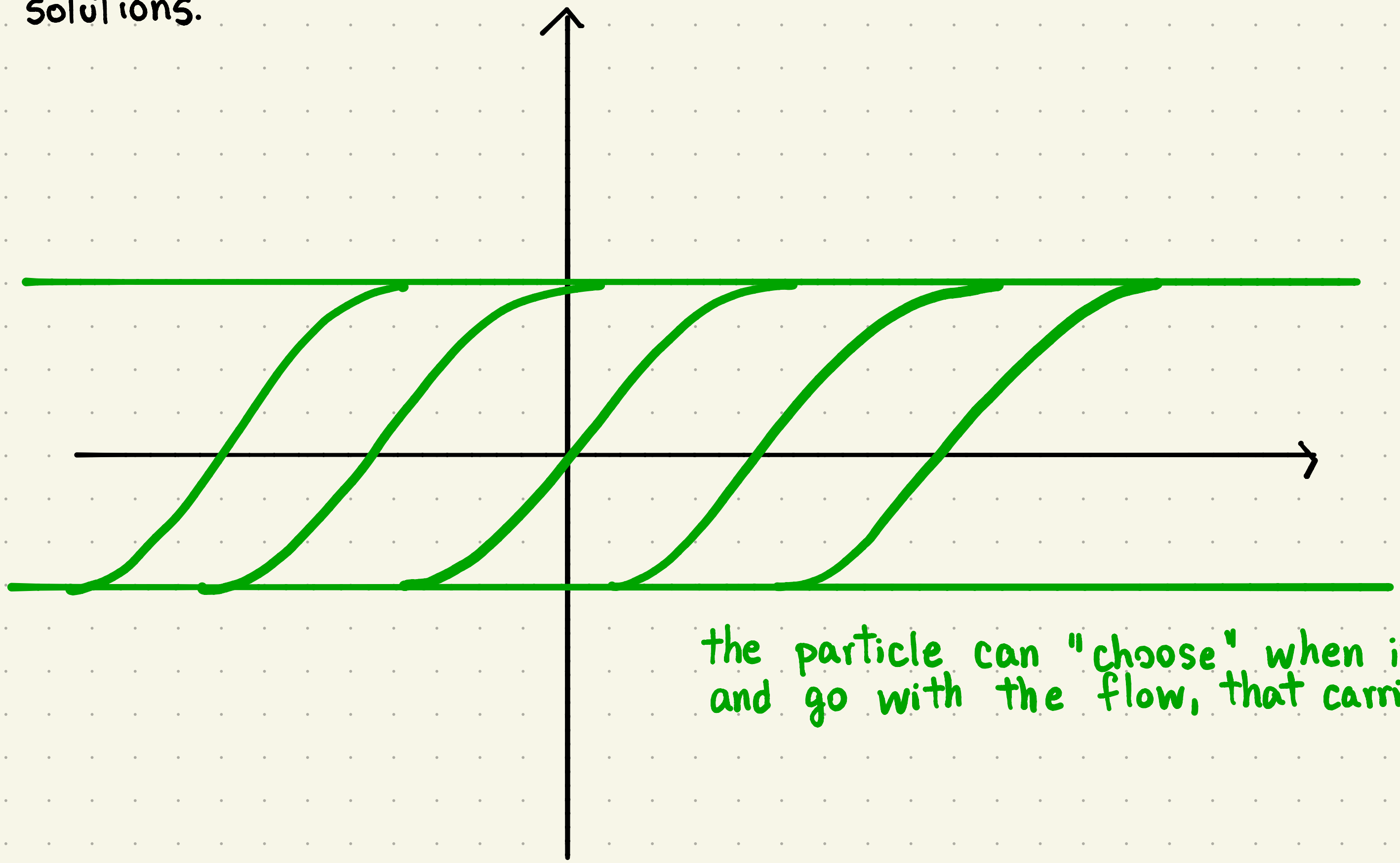
Remember the problem was

$$y' = \sqrt{1-y^2}, \quad y(a) = 1 \qquad ⊛$$

and we found solutions

$$y(t) = \begin{cases} -1 & , t+C < -\pi/2 \\ \sin(t+C) & , -\frac{\pi}{2} \leq t+C \leq \frac{\pi}{2} \\ +1 & , \frac{\pi}{2} \leq t+C \end{cases}$$

To match the condition $y(a)=1$, we only need $\frac{\pi}{2} \leq a+C \Rightarrow C \geq \frac{\pi}{2} - a$. So ⊛ has infinitely many solutions.



the particle can "choose" when it wants to leave -1 and go with the flow, that carries it to 1.

EXAMPLE: Consider the IVP

$$x^2 y' + y^2 = 0 \,, \quad y(a) = b.$$

For which values of a and b does the problem have an unique solution?

1) When does Picard's Theorem apply?

2) For those values of (a,b) where it does not apply, are there more solutions?

1) $x^2 y' + y^2 = 0 \Rightarrow y' = -\frac{y^2}{x^2} =: f(x,y)$ undefined at $x=0$.
$\frac{\partial f}{\partial y} = \frac{-2y}{x^2}$ undefined at $x=0$.
$\Rightarrow$ Unique solution when $a \neq 0$.

2) What if $a=0$?

There is the constant solution $y=0$. Can separate variables:

$$\frac{-y'}{y^2} = \frac{1}{x^2} \Rightarrow \frac{1}{y} = -\frac{1}{x} + C \Rightarrow \boxed{y = \frac{x}{-1+Cx}}$$

Check! $x^2\left(\frac{x}{-1+Cx}\right)' + \left(\frac{x}{-1+Cx}\right)^2 \stackrel{?}{=} 0$

All of those solutions satisfy $y(0) = 0$.

ANSWER: one solution if $a \neq 0$, no solutions if $a=0$ and $b \neq 0$, infinitely many solutions if $a=b=0$.