1.4 SEPARABLE EQUATIONS

Those are equations of the form

$$y'(x) = f(x)g(y)$$

METHOD OF SOLUTION:

$$\frac{dy}{dx} = f(x)g(y)$$

$$\frac{dy}{g(y)} = f(x)\,dx$$

$$\boxed{\int \frac{dy}{g(y)} = \int f(x)\,dx}$$ (IMPLICIT SOLUTION)

Solve for y to get an explicit solution.

The method of separation of variables yields a one-parameter family of solutions. A singular solution is a solution that is not included in the 1-parameter family yielded by the separation of variables method.

$y' = 2xy$

$$y'(x) = 2x\,y(x)$$

$$\frac{y'(x)}{y(x)} = 2x$$ ⟵ if $y(x) \neq 0$ (here we have missed the solution $y \equiv 0$)

$$\ln|y(x)| = x^2 + C$$

$$|y(x)| = e^C \cdot e^{x^2}$$ ⟵ notice that e^C is always positive

$$\boxed{y(x) = De^{x^2}}$$ ⟵ if $|y(x)| = e^C \cdot e^{x^2}$ then either $y(x) = e^C e^{x^2}$ or $y(x) = -e^C e^{x^2}$. As C varies, $\pm e^C$ can attain any nonzero value. By writing De^{x^2}, we cover the possibilities $\pm e^C e^{x^2}$ and 0.

$$x\frac{dy}{dx} = (1-2x^2)\tan y$$

$$\frac{dy}{\tan y} = \frac{1-2x^2}{x}$$ ⟵ we have missed the solution $y \equiv 0, \pm\pi, \pm 2\pi, \dots$

$$\int \frac{\cos y\, dy}{\sin y} = \int \frac{1}{x} - 2x\, dx = \ln|x| - x^2 + C$$

$\parallel$ ⟵ $\sin y = z$, $\cos y\, dy = dz$

$$\int \frac{dz}{z} = \ln|z| = \ln|\sin y|$$

$$\Longrightarrow \ln|\sin y| = \ln|x| - x^2 + C$$

$$|\sin y| = e^C |x| \cdot e^{-x^2}$$

$$\sin y = D x \cdot e^{-x^2}$$ ⟵ D covers the possibilities $\pm e^C x e^{-2x^2}$ and 0

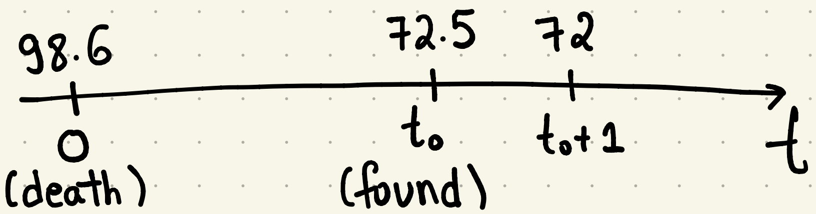
$$\boxed{y(x) = \sin^{-1}(D x \cdot e^{-x^2})}$$ ⟵ not all D yield solutions, because the argument of $\sin^{-1}$ has to be in $(-1,1)$. One has to choose the range of $\sin^{-1}$. Check the slope field!

In a room at 70°F, a murder occurred. The victim's body temperature was 98.6°F at the moment of death. When the body was found, its temperature was 72.5°F. One hour after the body was discovered, its temperature was 72°F. How long after death was the body discovered?

Assume NEWTON'S LAW OF COOLING: the rate of change of the temperature of a body is proportional to the difference between it's temperature and the ambient temperature.

<u>Solution</u>: let $T(t)$ be the temperature of the body (in °F) t hours after death.

<u>GOAL</u>: find t_0 such that $T(t_0) = 72.5$



<u>GIVEN</u>: $\dfrac{dT}{dt} = -k(70-T)$ for some unknown $k>0$

$$\circledast \begin{cases} T(0) = 98.6 \\ T(t_0) = 72.5 \\ T(t_0+1) = 72 \end{cases}$$

<u>PLAN</u>: 1) find the gen. sol. to $\dfrac{dT}{dt} = -k(70-T)$ (2 unknowns, C and k)

2) plug $\circledast$ into the gen. solution to find C, k and t_0.

1) Solve $\frac{dT}{dt} = -k(70 - T)$

$$\frac{dT}{70-T} = -k\,dt$$

$$\ln(T-70) = -kt + C$$ in our problem, $T > 70$ up to time t_0

$$T - 70 = e^c e^{-kt}$$

$$T(t) = 70 + e^c e^{-kt}$$

2) $T(0) = 98.6 \Rightarrow 98.6 = 70 + e^c \Rightarrow e^c = 28.6$

$T(t_0) = 72.5 \Rightarrow 70 + 28.6\, e^{-kt_0} = 72.5 \Rightarrow 28.6\, e^{-kt_0} = 2.5$

$T(t_0) - T(t_0+1) = 0.5 \Rightarrow 28.6\left(e^{-kt_0} - e^{-k(t_0+1)}\right) = 0.5$

$\Rightarrow 28.6\, e^{-kt_0}(1 - e^{-k}) = 0.5$

$\Rightarrow 2.5(1 - e^{-k}) = 0.5$

$\Rightarrow 1 - e^{-k} = 0.2 \Rightarrow e^{-k} = 0.8$

$28.6\, e^{-kt_0} = 2.5 \Rightarrow 28.6\,(0.8)^{t_0} = 2.5 \Rightarrow t_0 = \dfrac{\ln \frac{2.5}{28.6}}{\ln 0.8} = \underline{10.9}$

ANSWER: the body was found 10.9 hours after death.

CHECK ANSWER: we found $T(t) = 70 + 28.6\,(0.8)^t$ and $t_0 = 10.9$.

$T(0) = 98.6$ OK $\qquad T(t_0+1) = 72$ OK

$T(t_0) = 72.5$ OK

Consider the IVP $(y')^2 = 4y$, $y(a) = b$. When does it have no solutions? When does it have infinitely many solutions? When does it have a unique solution?

Since $(y')^2$ is always ≥ 0, there are no solutions when $b<0$.

If $b \geq 0$, we can write $y' = \pm 2\sqrt{y} =: f(t,y)$. Notice that $\frac{\partial f}{\partial y} = \frac{1}{\sqrt{y}}$ is not defined when $y=0$. The existence and uniqueness thm garantees unique solution when $b>0$. However this uniqueness is for the eq. $y' = 2\sqrt{y}$, and not for the eq. $(y')^2 = 2\sqrt{y}$. For the original eq. there are two solutions when $b>0$: one is the solution of $y' = 2\sqrt{y}$ and the other is the solution of $y' = -2\sqrt{y}$.

We can try to solve by separating the variables:

$$y' = \pm 2\sqrt{y} \Rightarrow \frac{dy}{2\sqrt{y}} = \pm 1$$

$$\Rightarrow \sqrt{y} = \pm x + C \Rightarrow y(x) = (\pm x + C)^2$$ two solutions when $b>0$.

Infinitely many when $b=0$.