1.5 FIRST ORDER LINEAR EQUATIONS

Diff. eqs. we can (sometimes) solve: $\frac{dy}{dx} = \frac{g(x)}{h(y)}$ (separable)

Today: $\frac{dy}{dx} + P(x)y = Q(x)$ (first-order linear)

First order: the equation has y' but does not have $y'', y''', \ldots$

$y' = 2xy$ 1st order

$y'' = -y$ 2nd order

Linear: if y_1 and y_2 solve the equation then so does any linear combination of y_1 and y_2 (i.e. $ay_1 + by_2$ for constants a and b). Verify this!

EXAMPLES:

LINEAR	NON LINEAR
$y' = 2xy$	$y' = 2xy^2$
$y' = y \cdot \cos x$	$y' = x \cdot \cos y$
$y'' = 1 + y'$	$y'' = \sqrt{1+(y')^2}$

THEOREM (EXISTENCE AND UNIQUENESS FOR LINEAR EQUATIONS)

The IVP $\frac{dy}{dx} + P(x)y = Q(x)$, $y(x_0) = y_0$

has one and only one solution provided the functions P and Q are continuous near (x_0, y_0).

The solution is defined for all x where P and Q are continuous.

Compare

$$y' = y, \quad y(0) = 1 \qquad \text{linear, solution defined in } (-\infty, \infty)$$

with

$$y' = y^2, \quad y(0) = 1 \qquad \text{nonlinear, solution defined in } (-\infty, 1)$$

METHOD OF SOLUTION FOR 1ST ORDER LINEAR EQS

$$\frac{dy}{dx} + P(x)\,y = Q(x)$$

1) Compute $\rho(x) := \exp\left(\int P(x)\,dx\right)$ "integrating factor"

2) Multiply the equation by the integrating factor

$$\frac{dy}{dx}\exp\left(\int P(x)dx\right) + P(x)\exp\left(\int P(x)\,dx\right) y = Q(x)\exp\left(\int P(x)\,dx\right)$$

3) Rewrite as

$$\frac{d}{dx}\left(y \cdot \exp\left(\int P(x)\,dx\right)\right) = Q(x)\exp\left(\int P(x)\,dx\right)$$

4) Integrate both sides (if you can...)

EXAMPLE: $\frac{dy}{dt} = \frac{3}{t} y + t^5$

This equation is NOT SEPARABLE!

But it is 1st order linear:

$y' - \frac{3}{t} y = t^5$

Integrating factor: $\exp\left(-\int \frac{3}{t} dt\right) = \exp(-3 \ln t) = t^{-3}$ ← no need for C

Multiply the equation by the integrating factor:

$$t^{-3} y' - 3 t^{-4} y = t^2$$

$$(t^{-3} y)' = t^2$$

$$t^{-3} y = \frac{t^3}{3} + C$$

$$\boxed{y = \frac{t^6}{3} + C t^3}$$

EXAMPLE: $y' + 2xy = x$

Notice that the eq. is both 1st order linear and separable. We can solve it in two different ways.

SOLUTION 1: integrating factor

$$y' + 2xy = x$$

Int. factor : $\exp\left(\int 2x\,dx\right) = e^{x^2}$

Multiply eq. by the int. factor:

$$e^{x^2} y' + 2x e^{x^2} y = x e^{x^2}$$

$$\left(e^{x^2} y\right)' = x e^{x^2}$$

$$e^{x^2} y = \int x e^{x^2} dx = \frac{e^{x^2}}{2} + C$$

$$\boxed{y = \frac{1}{2} + Ce^{-x^2}}$$

SOLUTION 2: separation of variables

$$\frac{dy}{dx} + 2xy = x$$

$$dy = x(1-2y)\,dx$$

$$\int \frac{dy}{1-2y} = \int x\,dx = \frac{x^2}{2} + C$$

we have missed the solution $y = \frac{1}{2}$

$$\frac{-1}{2} \ln|1-2y| = \frac{x^2}{2} + C$$

$$1-2y = \pm \exp\left(-x^2 + C\right)$$

new $C = -2$ (previous C)

$$1-2y = Ce^{-x^2}$$

covers the possibilities $\pm e^c$ and $c=0$, that is $y = \frac{1}{2}$, which we missed when dividing by $\frac{1}{1-2y}$

$$\boxed{y = \frac{1}{2} + Ce^{-x^2}}$$

EXAMPLE: Regarding x as a function of y, solve the equation

$$(1-4xy^2)\frac{dy}{dx} = y^3$$

Notice that if we try to solve for y the eq. is not separable and not linear. However if we try to solve for x then it becomes linear.

$$(1-4xy^2)\frac{dy}{dx} = y^3 \Rightarrow \frac{dx}{dy} = \frac{1-4xy^2}{y^3} \Longrightarrow \frac{dx}{dy} + \frac{4}{y}x = \frac{1}{y^3} \quad (\mathrm{I})$$

Integrating factor: $\exp\left(\int \frac{4}{y}dy\right) = \exp(4\ln y) = y^4$

Multiply (I) by the integrating factor:

$$y^4\frac{dx}{dy} + 4y^3x = y$$

$$\frac{d}{dy}(y^4x) = y$$

$$x = y^{-4}\int y\,dy = y^{-4}\left(\frac{y^2}{2}+C\right)$$

$$\boxed{x = \frac{1}{2y^2} + \frac{C}{y^4}}$$

EXAMPLE: $y' + y = 2\sin x$

Look at the slope field!

Integrating factor: $\exp\left(\int 1\,dx\right) = e^x$

Multiply the equation by the integrating factor:

$$e^x y' + e^x y = 2e^x \sin x$$

$$(e^x y)' = 2e^x \sin x$$

$$e^x y = \int 2e^x \sin x\,dx \quad \text{(I)}$$

Can integrate by parts twice:

$$\int 2e^x \sin x\,dx = \int 2\sin x\,d(e^x) = 2e^x \sin x - \int 2e^x \cos x\,dx$$

$$= 2e^x \sin x - \left[2e^x \cos x - \int 2e^x(-\sin x)\,dx\right]$$

$$\Rightarrow 4\int e^x \sin x\,dx = 2e^x(\sin x - \cos x)$$

Back to (I):

$$e^x y = e^x(\sin x - \cos x) + C$$

$$\boxed{y = \sin x - \cos x + Ce^{-x}}$$

EXAMPLE: regarding x as a function of y, solve the equation

$$(1+2xy)\frac{dy}{dx} = 1+y^2$$

Try plotting the slope field. Solving for x instead of y amounts to parametrizing the curves on a different coordinate system. What is the advantage of doing this? If y is the dependent variable, the eq. is NOT LINEAR nor separable. If x is the dependent variable, then the eq. is linear, therefore solvable.

$$(1+2xy)\frac{dy}{dx} = 1+y^2 \Rightarrow (1+y^2)\frac{dx}{dy} = 1+2xy \Longrightarrow \frac{dx}{dy} - \frac{2y}{1+y^2}x = \frac{1}{1+y^2} \qquad (\mathrm{I})$$

Integrating factor: $\exp\left(\int \frac{-2y}{1+y^2}dy\right) \underset{\substack{1+y^2=u\\ 2y\,dy=du}}{=} \exp\left(\int -\frac{du}{u}\right) \underset{\substack{u \text{ is always } \geq 0,\\ \text{so no abs. value needed}}}{=} \exp\left(-\ln(1+y^2)\right) = \frac{1}{1+y^2}.$

Multiply (I) by the integrating factor:

$$\left(\frac{1}{1+y^2}\right)\frac{dx}{dy} - \frac{2y}{(1+y^2)^2}x = \frac{1}{(1+y^2)^2}$$

$$\frac{d}{dy}\left(\frac{x}{1+y^2}\right) = \frac{1}{(1+y^2)^2}$$

$$\Rightarrow x = (1+y^2)\int\frac{dy}{(1+y^2)^2} \qquad (\mathrm{II})$$

The integral $\int \frac{dy}{(1+y^2)^2}$ is tricky. There are two ways of doing it.

A) Trigonometric substitution

$$\int\frac{dy}{(1+y^2)^2} \underset{\substack{y=\tan\theta\\ dy=(1+\tan^2\theta)d\theta\\ =(1+y^2)d\theta}}{=} \int\frac{(1+y^2)\,d\theta}{(1+y^2)(1+\tan^2\theta)} \underset{\tan\theta=\frac{\sin\theta}{\cos\theta}}{=} \int \cos^2\theta\, d\theta$$

$$\int \cos^2\theta - \sin^2\theta \, d\theta = \int \cos 2\theta \, d\theta = \frac{1}{2}\sin 2\theta + C$$

$$\int \cos^2\theta + \sin^2\theta \, d\theta = \int 1 \, d\theta = \theta + C$$

$$\Longrightarrow \int \cos^2\theta \, d\theta = \frac{1}{2}\theta + \frac{1}{4}\sin 2\theta + C$$

Back to the original integral

$$\int \frac{dy}{(1+y^2)^2} \underset{y=\tan\theta}{=} \int \cos^2\theta \, d\theta = \frac{1}{2}\theta + \frac{1}{2}\sin\theta\cdot\cos\theta + C = \frac{1}{2}\tan^{-1}y + \frac{y}{2(1+y^2)} + C$$

$y = \tan\theta$

$1 + y^2 = \cos^2\theta$

$\Longrightarrow \sin\theta\cos\theta = \frac{y}{1+y^2}$

B) Recursion + integration by parts

$$\frac{1}{(1+y^2)^2} = \frac{1}{1+y^2} - \frac{y^2}{(1+y^2)^2} \Longrightarrow \int \frac{dy}{(1+y^2)^2} = \int \frac{dy}{1+y^2} - \int \frac{y}{2}\, d\left(-\frac{1}{1+y^2}\right)$$

$$= \tan^{-1}y - \left[\frac{y}{2}\left(-\frac{1}{1+y^2}\right) - \int\left(-\frac{1}{1+y^2}\right)\frac{dy}{2} \right]$$

$$= \frac{1}{2}\tan^{-1}y + \frac{y}{2(1+y^2)} + C$$

Back to the differential equation (II):

$$x = (1+y^2)\int \frac{dy}{(1+y^2)^2} = (1+y^2)\left[\frac{1}{2}\tan^{-1}y + \frac{y}{2(1+y^2)} + C\right]$$

$$\boxed{x = \frac{1}{2}y + \frac{1}{2}(1+y^2)\tan^{-1}y + C(1+y^2)}$$