1.5B FIRST-ORDER LINEAR EQUATIONS

MIXING PROBLEM 1: A vat contains 60L of water mixed with 5kg of salt. A salt water solution that contains 2kg of salt per liter enters the tank at a constant rate of 3L/min. Pure water is also flowing into the vat at a rate of 2L/min. The solution is kept well mixed and leaves the vat at a rate of 5L/min. How much salt remains after 30 minutes? What is the long term behavior?

what is the goal? If $s(t)$ denotes the amount of salt (in kg) after t minutes, then we are looking for $s(30)$.

what information do we have?

$s(0) = 5$ $\qquad s'(t) = 2 \cdot 3 - 5 \cdot \frac{s(t)}{60}$ ← the vat always has 60L of water

PLAN: solve the 1st-order linear equation for $s(t)$ (there will be a free parameter C), use $s(0) = 5$ to solve for C and compute $s(30)$.

$s' + \frac{s}{12} = 6$ $\qquad$ Integrating factor $\exp\left(\int \frac{dt}{12}\right) = \exp\left(\frac{t}{12}\right)$

$$\Rightarrow e^{t/12}s' + \frac{e^{t/12}s}{12} = 6e^{t/12} \Rightarrow (e^{t/12}s)' = 6e^{t/12} \Rightarrow e^{t/12}s = 72e^{t/12} + C$$

$$\Rightarrow s(t) = 72 + Ce^{-t/12}$$

$s(0) = 5 \Rightarrow 5 = 72 + C \Rightarrow C = -67 \Rightarrow \boxed{s(t) = 72 - 67e^{-t/12}}$ $\qquad s(30) = 66.5$

ANSWER: after 30 min, there will be 66.5kg of salt in the tank. The amount of salt approaches 72kg. Draw the graph.

$$y' = 6 - y/12$$

$$y = 2 + Ce^{-t/12}$$

$$y' = -\frac{C}{12} e^{-t/12}$$

$$y' + \frac{y}{12} = -\frac{C}{12} e^{-t/12} + \frac{2}{12} + \frac{C}{12} e^{-t/12}$$

MIXING PROBLEM 2: - A 400-gallon tank contains 200 gallons of water with 3 pounds of salt per gallon. Water flows into the tank at 6 gallons per minute, containing 5 pounds of salt per gallon. Water flows out of the tank at 4 gallons per minute. How much salt is there in the tank when it is full ?

When is the tank full? After 100 min. $\leftarrow \frac{200\,\text{gal}}{2\,\text{gal/min}}$

$s(t)$ = pounds of salt after t min WANT: $s(100)$

KNOW: $s(0) = 600$, $s'(t) = 6 \cdot 5 - 4 \cdot \frac{s(t)}{200+2t}$ ← volume of water in the tank at time t

PLAN: solve the eq. for $s(100)$.

$$s' + \frac{2}{100+t} s = 30$$

Int. factor $\exp\left(\int \frac{2\,dt}{100+t}\right) = \exp\left(2 \cdot \ln(100+t)\right) = (100+t)^2$

Multiply: $(100+t)^2 s' + 2(100+t)s = 30(100+t)^2 \Rightarrow ((100+t)^2 s)' = 30 \cdot (100+t)^2$

$$\Rightarrow (100+100)^2 s(100) - (100+0)^2 s(0) = \int_0^{100} 30(100+t)^2\,dt = 10(200^3 - 100^3)$$

$$\Rightarrow s(100) = 200^{-2}\left[10(200^3 - 100^3) + 100^2 \cdot 600\right]$$

$$= 100(200 - 50) + 150$$

$$= 1800$$