1.6 EXACT EQUATIONS

Exact equations have the form

$$M(x,y)\,dx + N(x,y)\,dy = 0, \quad \text{where } \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

They have implicit solutions of the form

$$F(x,y) = C, \qquad \text{where } \frac{\partial F}{\partial x} = M \text{ and } \frac{\partial F}{\partial y} = N.$$

Check: if $\frac{\partial F}{\partial x} = M$, $\frac{\partial F}{\partial y} = N$ and $F(x,y) = C$, then

$$\frac{d}{dx} F(x, y(x)) = 0$$

$$\Longrightarrow \underbrace{\frac{\partial F}{\partial x}(x, y(x))}_{M} + \underbrace{\frac{\partial F}{\partial y}(x, y(x))}_{N} \frac{dy}{dx} = 0.$$

Also, $\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\frac{\partial F}{\partial x} = \frac{\partial}{\partial x}\frac{\partial F}{\partial y} = \frac{\partial N}{\partial y}.$

EXAMPLE: $\frac{x}{2}\cdot \cot y \cdot \frac{dy}{dx} = -1$

$\frac{2}{x}\,dx + \frac{1}{2}\cot y\; dy = 0$ ⟵ Exact, because $\frac{\partial}{\partial y}\frac{1}{x} = \frac{\partial}{\partial x}\cot y$

We look for a function $F(x,y)$ such that $\frac{\partial F}{\partial x} = \frac{2}{x}$ and $\frac{\partial F}{\partial y} = \cot y$.

$\frac{\partial F}{\partial x} = \frac{2}{x} \Rightarrow F(x,y) = \ln(x^2) + C(y)$ ⟵ any function of y has $\frac{\partial}{\partial x}C(y) = 0$

$\frac{\partial F}{\partial y} = \cot y \Rightarrow \frac{\partial}{\partial y}\left(\ln(x^2) + C(y)\right) = \cot y$

$\Rightarrow C'(y) = \cot y$

$\Rightarrow C(y) = \int \frac{\cos y}{\sin y}\,dy = \ln|\sin y| + C$ ⟵ we could have written $C(x)$, but then would need $\frac{\partial}{\partial x}C(x) = \frac{1}{x}$, repeating the 1st step

$F(x,y) = \ln(x^2) + \ln|\sin y| + C = \ln|x \sin y| + C$

Implicit solution to $\frac{dx}{x} + \frac{1}{2}\cot y\; dy = 0$: $\boxed{x^2 \sin y = C}$ graph this!

EXAMPLE: $e^y\,dx + (xe^y+2y)\,dy = 0$

Is the equation exact?

$$\frac{\partial}{\partial y} e^y \overset{?}{=} \frac{\partial}{\partial x}(xe^y+2y)$$

We look for $F(x,y)$ such that $\frac{\partial F}{\partial x} = e^y$ and $\frac{\partial F}{\partial y} = xe^y + 2y$.

$$\frac{\partial F}{\partial y} = xe^y+2y \Rightarrow F(x,y) = xe^y + y^2 + C(x) \quad \text{(I)}$$

$$\frac{\partial F}{\partial x} = e^y \Rightarrow \frac{\partial}{\partial x}(xe^y + y^2 + C(x)) = e^y$$

$$\Rightarrow e^y + C'(x) = 0$$

$$\Rightarrow C'(x) = 0 \Rightarrow C(x) \overset{\text{(II)}}{=} C$$

this means that $C(x)$ can be any constant function

It follows from (I) and (II) that

$$F(x,y) = xe^y + y^2 + C,$$

so the equation $e^y\,dx + (xe^y+2y)\,dy = 0$ has implicit solution

$$\boxed{xe^y + y^2 = C}$$

graph!

EXAMPLE: $(x\cos x + e^y)dx + xe^y dy = 0$

Is it exact?

$$\frac{\partial}{\partial y}(x\cos x + e^y) \stackrel{?}{=} \frac{\partial}{\partial x}(xe^y)$$

Look for $F(x,y)$ such that $\frac{\partial F}{\partial x} = x\cos x + e^y$ and $\frac{\partial F}{\partial y} = xe^y$.

Can start with $\frac{\partial F}{\partial x} = x\cos x + e^y \Rightarrow F(x,y) = \int x\cos x + e^y\, dx$ (I)

but it is easier to start with $\frac{\partial F}{\partial y} = xe^y \Rightarrow F(x,y) = \int xe^y\, dy = \underset{(\text{II})}{xe^y} + C(x)$ ← this integral is simpler than (I)

$$\frac{\partial F}{\partial x} = x\cos x + e^y \Rightarrow \frac{\partial}{\partial x}(xe^y + C(x)) = x\cos x + e^y$$

$$\Rightarrow e^y + C'(x) = x\cos x + e^y$$

$$\Rightarrow C(x) = \int x\cos x\, dx$$ ← couldn't avoid this integral

$$= \int x\, d(-\sin x)$$

$$= -x\sin x - \int(-\sin x)dx$$

$$= -x\sin x - \cos x + C$$

$$\Rightarrow F(x,y) \stackrel{(\text{II})}{=} xe^y - x\sin x - \cos x + C$$

The equation $(x\cos x + e^y)dx + xe^y dy = 0$ has the implicit solution

$$\boxed{x(e^y - \sin x) - \cos x = C}$$

EXAMPLE: the eq. $y' + 2xy = x$ (1st order linear) is NOT exact, but becomes exact after being multiplied by the integrating factor.

$$\frac{dy}{dx} + x(2y-1) = 0$$

$$\Longrightarrow x(2y-1)dx + dy = 0 \qquad (*)$$

$$\frac{\partial}{\partial y}[x(2y-1)] \neq \frac{\partial}{\partial y}[1] \qquad \text{NOT EXACT}$$

Integrating factor : $\exp(\int 2x\,dx) = e^{x^2}$

Multiply $(*)$ by the integrating factor:

$$(2y-1)xe^{x^2}dx + e^{x^2}dy = 0$$

$$\frac{\partial}{\partial y}[(2y-1)xe^{x^2}] = \frac{\partial}{\partial x}e^{x^2} \qquad \text{EXACT}$$