1.6B Substitution methods and homogeneous equations

In calculus, the integrals can be divided in three classes

1) Easy integrals and their combinations: $\int x^2 dx$, $\int \sin y\, dy$, $\int e^u du$, $\int \frac{dx}{1+x^2}$, etc

2) Integrals that can be converted into an easy integral by a substitution, integration by parts, partial fractions... such as $\int \tan y\, dy$, $\int x \cos x\, dx$, $\int \ln x\, dx$, $\int \frac{x^2}{\sqrt{1-x^2}} dx$, etc.

3) All other integrals. These have to be computed numerically.

Analogously, we can classify the differential equations in

1) Equations for which there exists a method of solution: separable, exact, linear...

2) Equations that can be transformed into an equation of the first type by rearranging, change of variables, integrating factor...

3) All other equations.

Section 1.6 deals with equations of the second type.

EXAMPLE $xy'' - y' = 3x^2$ $\qquad y = x^3 + Cx^2 + D$

Notice that there is no y in the eq. Can solve for y' first, then integrate to find y.

If y' is the unknown function, the eq. is 1st order linear:

$$y'' - \frac{1}{x}y' = 3x$$

Int. factor $\exp\left(\int -\frac{1}{x}dx\right) = \exp(-\ln x) = \frac{1}{x}$

Multiply: $\quad \frac{1}{x}y'' - \frac{1}{x^2}y' = 3$

$$\left(\frac{1}{x}y'\right)' = 3$$

$$\frac{1}{x}y' = 3x + C$$

$$y'(x) = 3x^2 + Cx$$

$$\boxed{y(x) = x^3 + \frac{Cx^2}{2} + D}$$

← different integration, different constant!

EXAMPLE: solve $x\frac{dy}{dx} - 4x^2 y + 2y \ln y = 0$ by means of the substitution $v = \ln y$.

$v = \ln y \qquad dv = \frac{dy}{y}$

$$x\frac{dy}{dx} - 4x^2 y + 2y \ln y = 0$$

$$x\frac{e^v dv}{dx} - 4x^2 e^v + 2e^v v = 0$$

Cancel e^v, divide by x:

$\frac{dv}{dx} + \frac{2}{x}v = 4x$ 1st order linear, can solve with integrating factor

$v = x^2 + \frac{C}{x^2}$

Back to $v = \ln y$

$\ln y = x^2 + \frac{C}{x^2}$

$$\boxed{y = \exp\left(x^2 + \frac{C}{x^2}\right)}$$

HOMOGENEOUS EQUATIONS

A differential equation of the form $y' = f(x,y)$ is HOMOGENEOUS if $f(ax,ay) = f(x,y)$ for every $a > 0$.

EXAMPLES

HOMOGENEOUS	NOT HOMOGENEOUS
$(x+y)dx - (x-y)dy = 0$	$(x+y)dx - (x-y+1)dy = 0$
$xy' = 2x + 3y$	$x^2 y' = 2x + 3y$
$\frac{dy}{dx} = \frac{x^2+xy+y^2}{x^2}$	$\frac{dy}{dx} = \frac{x^2+xy+y^3}{x^2}$
$y' = \frac{y}{x} + \frac{x}{y}$	$xy' = \frac{y}{x} + \frac{x}{y}$

HOW TO SOLVE HOMOGENEOUS EQUATIONS

Let $v = y/x$, change variables from (x,y) to (x,v). In the (x,v) variables, the eq. becomes separable.

EXAMPLE: $y' = \frac{y}{x} + \frac{x}{y}$ ⊛

Let $v = \frac{y}{x}$. Then $\frac{dy}{dx} = \frac{d}{dx}(xv) = x\frac{dv}{dx} + v$.

Rewrite ⊛ in terms of (x,v):

$$x\frac{dv}{dx} + v = v + \frac{1}{v} \implies \frac{dv}{dx} = \frac{1}{xv} \quad \text{separable}$$

Solve for v:

$$v\,dv = \frac{dx}{x} \implies \frac{v^2}{2} = \ln|x| + C \implies v = \pm\sqrt{\ln(x^2) + C}$$

Revert to the (x,y) variables:

$$\frac{y}{x} = \pm\sqrt{\ln(x^2)+C} \implies \boxed{y = \pm x\sqrt{\ln(x^2)+C}}$$

EXAMPLE: $(x+y)\,dx-(x-y)\,dy=0$

Is it exact? $\frac{\partial}{\partial y}(x+y) \overset{?}{=} \frac{\partial}{\partial x}[-(x-y)]$ NOT EXACT

But it is homogeneous: $\frac{dy}{dx}=\frac{x+y}{x-y}=\frac{ax+ay}{ax-ay}$ for any $a>0$.

Since it's homogeneous, the substitution $v=\frac{y}{x}$ turns it into a separable eq.

$y=xv \Rightarrow \frac{dy}{dx}=x\frac{dv}{dx}+v$

Rewrite the eq. in terms of (x,v): $x\frac{dv}{dx}+v=\frac{x+xv}{x-xv}$

Solve for v:

$$x\cdot\frac{dv}{dx}=\frac{1+v}{1-v}-v=\frac{1+v^2}{1-v} \implies \frac{1-v}{1+v^2}dv=\frac{dx}{x}$$

$$\Rightarrow \int\frac{dv}{1+v^2}-\int\frac{v}{1+v^2}dv=\ln|x|+C$$

$$\Rightarrow \arctan v-\frac{1}{2}\ln(1+v^2)=\ln|x|+C \qquad \text{(implicit solution)}$$

Revert to (x,y) variables:

$$\arctan\frac{y}{x}-\frac{1}{2}\ln\left(1+\frac{y^2}{x^2}\right)=\ln|x|+C \qquad \text{(implicit)}$$

Can simplify further:

$$\boxed{\arctan\frac{y}{x}-\ln\sqrt{x^2+y^2}=C}.$$