1.6 C Homogeneus & Bernoulli equations

A diff. eq. of the form $y' = f(x,y)$ is called HOMOGENEOUS if $f(Cx, Cy) = f(x,y)$ for all $C > 0$.

The change of variables $v = y/x$ turns a homogeneous eq. into a separable equation

$y = xv \Rightarrow y' = xv' + v$

$$xv' + v = f(1, v)$$

$$\Rightarrow \frac{dv}{f(1,v) - v} = \frac{dx}{x}$$

EXAMPLE: $\frac{dy}{dx} = \frac{x^2+xy+y^2}{x^2}$

Let $v = y/x$. Then $y = xv$ and $\frac{dy}{dx} = x\frac{dv}{dx} + v$.

Rewrite the eq. in the variables x and v:

$$x\frac{dv}{dx} + \cancel{v} = \frac{x^2+x^2v+x^2v^2}{x^2} = 1 + \cancel{v} + v^2 \qquad \text{(separable)}$$

Solve for $v(x)$:

$$\frac{dv}{1+v^2} = \frac{dx}{x} \Longrightarrow \arctan v = \ln|x| + C \Longrightarrow v = \tan(\ln|x|+C).$$

Revert to the (x,y) variables:

$$\boxed{y = x \cdot \tan(\ln|x| + C)}$$

EXAMPLE: $(x+y)dy - (x-y)dx = 0$ is both exact and homogeneous.

Let $v = \frac{y}{x}$, $y = xv$, $\frac{dy}{dx} = x\frac{dv}{dx} + v$.

Rewrite in terms of x and v:

$$(x + xv)\left[x\frac{dv}{dx} + v\right] - (x - xv) = 0$$ separable

Solve for v

$$(1+v)x\frac{dv}{dx} + (1+v)v = 1 - v$$

$$(1+v)x\frac{dv}{dx} = -v^2 - 2v + 1$$

$$\frac{dx}{x} = \frac{1+v}{-v^2-2v+1}dv = \frac{v+1}{-(v+1)^2+2}dv$$

$$\ln|x| = \int \frac{v+1}{-(v+1)^2+2}dv = -\frac{1}{2}\ln\left|-(v+1)^2+2\right| + c$$

$$|x| = e^c\left|-(v+1)^2+2\right|^{-1/2}$$

$$x^2(v^2 + 2v - 1) = C$$

Revert to the old variables:

$$x^2\left(\frac{y^2}{x^2} + 2\frac{y}{x} - 1\right) = C$$

$$\boxed{y^2 + 2xy - x^2 = C}.$$

EXAMPLE: $\frac{dy}{dx} = \frac{x^2+2xy+y^2}{x^2}$

Let $v = y/x$, $\frac{dv}{dx} = x\frac{dv}{dx} + v$. Rewrite the eq. in terms of x and v:

$$x\frac{dv}{dx} + v = (1+v)^2$$

Solve for v (separable):

$$\frac{dv}{1+v+v^2} = \frac{dx}{x}$$

$$\ln|x| = \int \frac{dv}{1+v+v^2} = \int \frac{dv}{(v+\frac{1}{2})^2 + \frac{3}{4}} = \frac{2}{\sqrt{3}} \cdot \arctan\left(\frac{2}{\sqrt{3}}\left(v+\frac{1}{2}\right)\right) + C$$

$$\int \frac{du}{u^2+a^2} = \frac{1}{a}\arctan\left(\frac{u}{a}\right)$$

$$\frac{\sqrt{3}}{2}\tan\left(\frac{\sqrt{3}}{2}\ln|x| + C\right) - \frac{1}{2} = v = \frac{y}{x}$$

$$\boxed{y = -\frac{x}{2} + x\frac{\sqrt{3}}{2}\tan\left(\frac{\sqrt{3}}{2}\ln|x| + C\right)}.$$

BERNOULLI EQUATIONS

A class of nonlinear equations that can be linearized by a change of variables.

$$\frac{dy}{dx} + P(x)y = Q(x)y^n \quad , \quad n \neq 0, 1$$ ⟵ linear when $n=0$ or 1

Linearized by the change of variables $v = y^{1-n}$.

EXAMPLE: $t^2y' + 2ty - y^3 = 0 \quad , \quad t > 0$

Rearrange as a Bernoulli equation:

$$y' + \frac{2}{t}y = \frac{1}{t^2}y^3$$ ⟵ $P(t) = \frac{2}{t}$, $Q(t) = \frac{1}{t^2}$, $n = 3$

Let $v = y^{1-3} = y^{-2}$. Then $y = (y^{-2})^{-1/2} = v^{-1/2}$ and $y' = -\frac{1}{2}v^{-3/2}v'$

Rewrite eq. in terms of t and v.

$$-\frac{1}{2}v^{-3/2}v' + \frac{2}{t}v^{-1/2} = \frac{1}{t^2}v^{-3/2}$$

Multiply by $-2v^{3/2}$:

$$v' - \frac{4}{t}v = \frac{-2}{t^2} \qquad \text{(linear)}$$

Can solve with int. factor. Solution: $v = \frac{2}{5t} + ct^4$.

Change back to (x,y):

$$y^{-2} = \frac{2}{5t} + ct^4 \Rightarrow \boxed{y = \left(\frac{2}{5t} + ct^4\right)^{-1/2}}$$