2.1 2.2 Population models, equilibrium solutions and stability

GOAL: predict the number of individuals in a population (bacteria in a Petri dish, fish in a pond, rabbits in a wood, people in a country...) by coming up with a function $P(t)$ that matches observations.

MODEL 1: EXPONENTIAL GROWTH

$P' = kP$ for some $k>0$ (a parameter of the model) that has to be determined experimentally

Fits data for bacterial population in a controlled environment with unlimited food.

What is the intuition behind? Each individual generates k offspring per unit of time, on average.

MODEL 2: LOGISTIC EQUATION

$P' = (a - bP)P$ for some $a, b > 0$

Idea: the average offspring per capita is basically constant when the population is small but decreases when the population increases.

Not realistic, but predicts that the population grows exponentially when small, then the growth slows down, then the population converges to a certain value.

MODEL 3: EXTINCTION-EXPLOSION

$P' = (-a+bP)P$ for some $a,b>0$

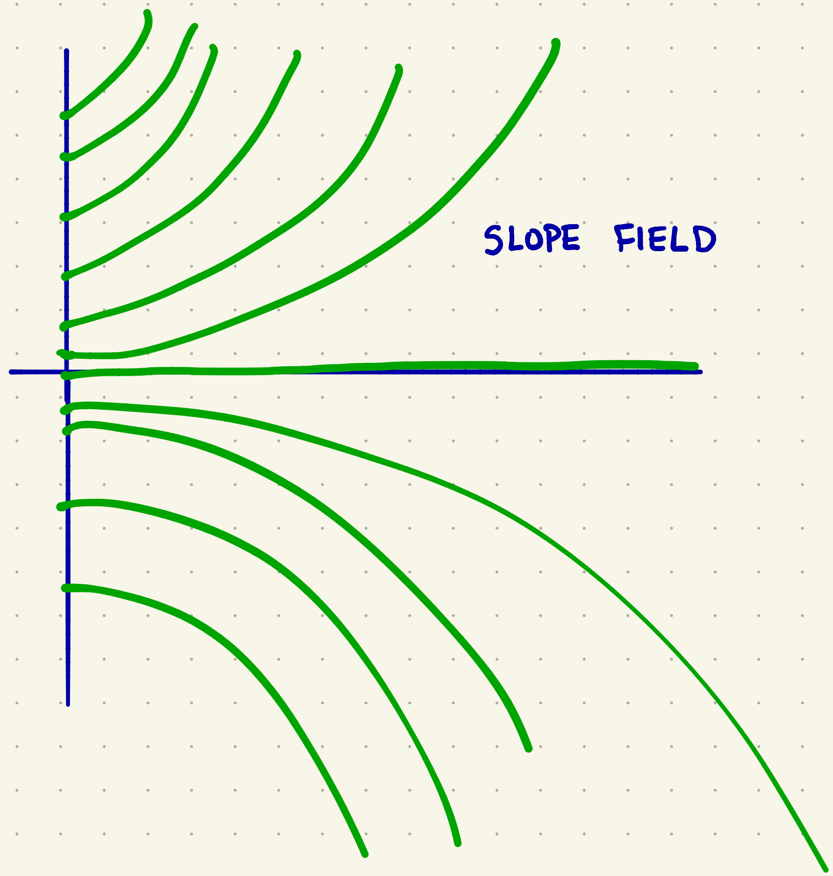
Idea: births occur whenever two individuals meet.

QUALITATIVE STUDY OF DIFFERENTIAL EQUATIONS

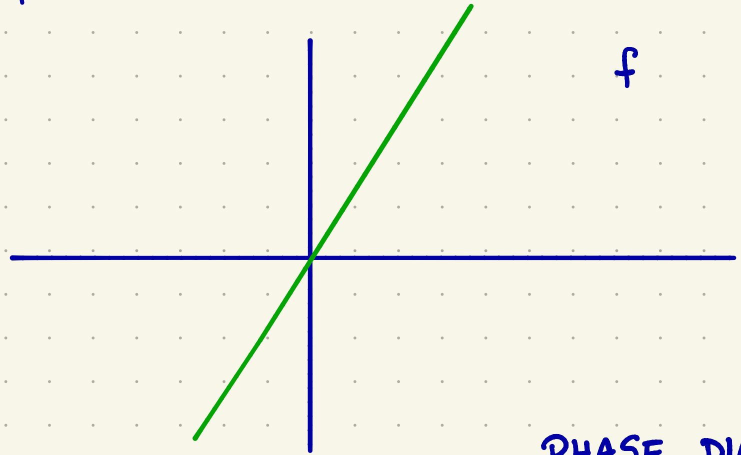
Consider $p' = f(p)$. What can we say about p WITHOUT SOLVING THE EQUATION?

- → equilibrium solutions are the zeros of f
- → any solution increases or decreases to an equilibrium or to $\pm\infty$

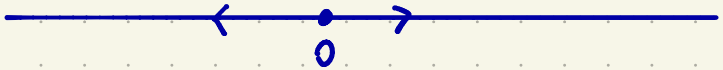
EXPONENTIAL $f(p) = kp$



SLOPE FIELD

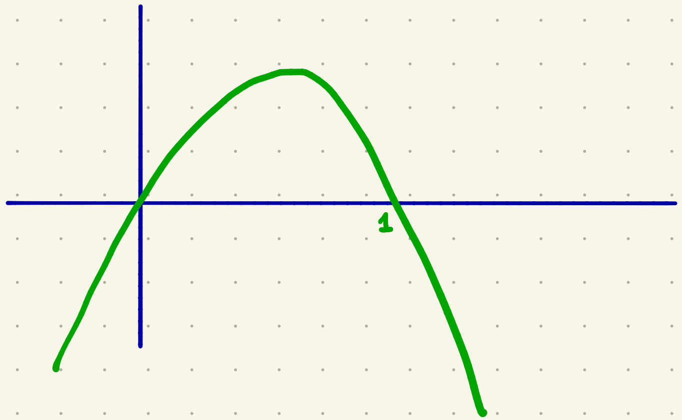


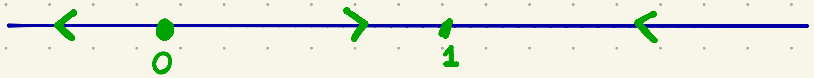
PHASE DIAGRAM



LOGISTIC EQUATION

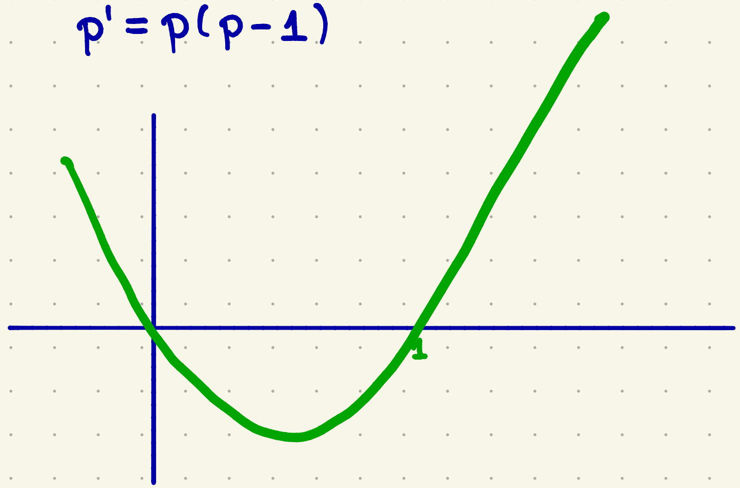
$p' = p(1-p)$

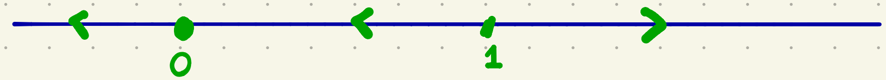




EXPLOSION - EXTINCTION

$p' = p(p-1)$





CLASSIFICATION OF EQUILIBRIUM SOLUTIONS

$y' = f(p)$ is called an autonomous equation if f does not depend on t

A number p_* is a critical point of the equation if $f(p_*) = 0$

If p_* is a critical point, the constant function $p(t) = p_*$ is called an equilibrium solution.

A critical point is said to be stable if solutions that start close to it converge to it.

A critical point is said to be unstable if it is not stable.

EXERCISE: classify the critical values in all of the previous examples.

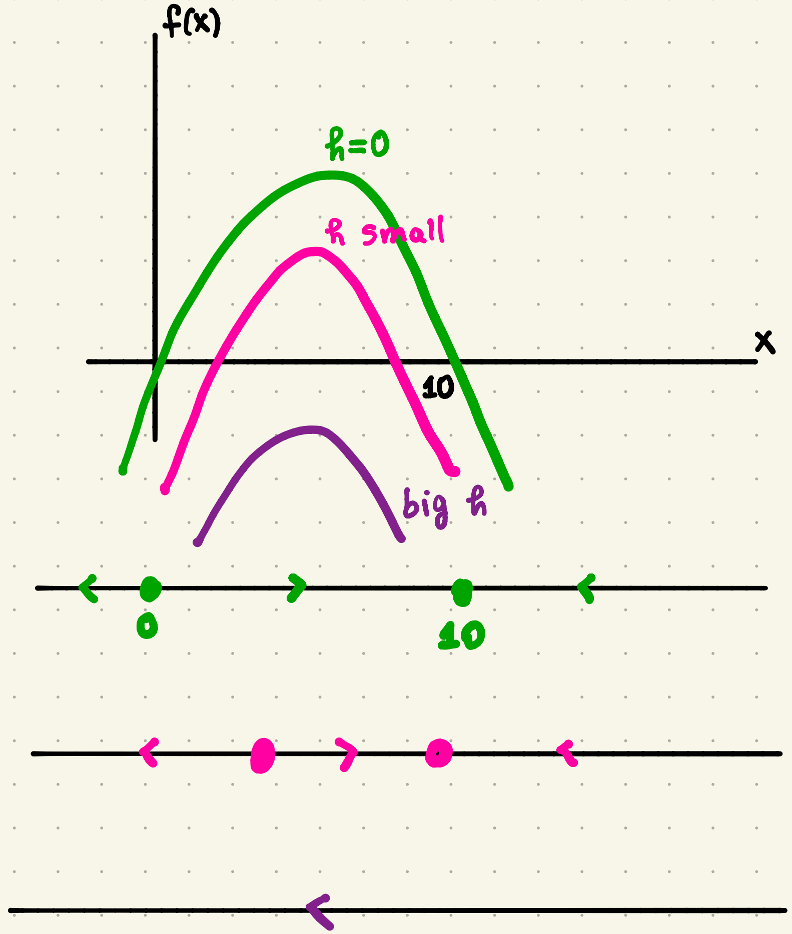
PROBLEM (LOGISTIC POPULATION WITH HARVEST)

Consider the autonomous equation

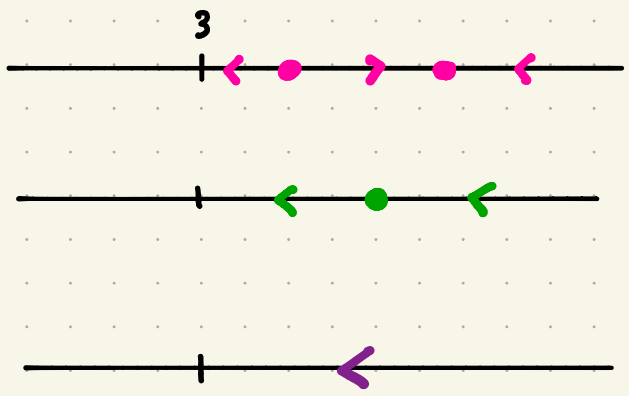
$$\frac{dx}{dt} = \frac{1}{10}x(10-x) - h$$
$$x(0) = 3$$

where $h > 0$. For which values of h does the solution eventually reaches zero?

Let us first sketch the graphs of $f(x) - h := \frac{1}{10}x(10-x) - h$ and the phase diagram of the equation $x' = f(x) - h$.



We are looking for h that makes one of these phase diagrams:



In other words, the smallest solution of $\frac{1}{10}x(10-x)-h=0$ is >3.

What are the solutions of $\frac{1}{10}x(10-x)-h=0$?

$$\frac{1}{10}x(10-x)-h=0$$

$$x^2-10x+10h=0$$

$$x=\frac{1}{2}\left(10\pm\sqrt{100-40h}\right)$$

So we need $\frac{1}{2}\left(10-\sqrt{100-40h}\right)>3$

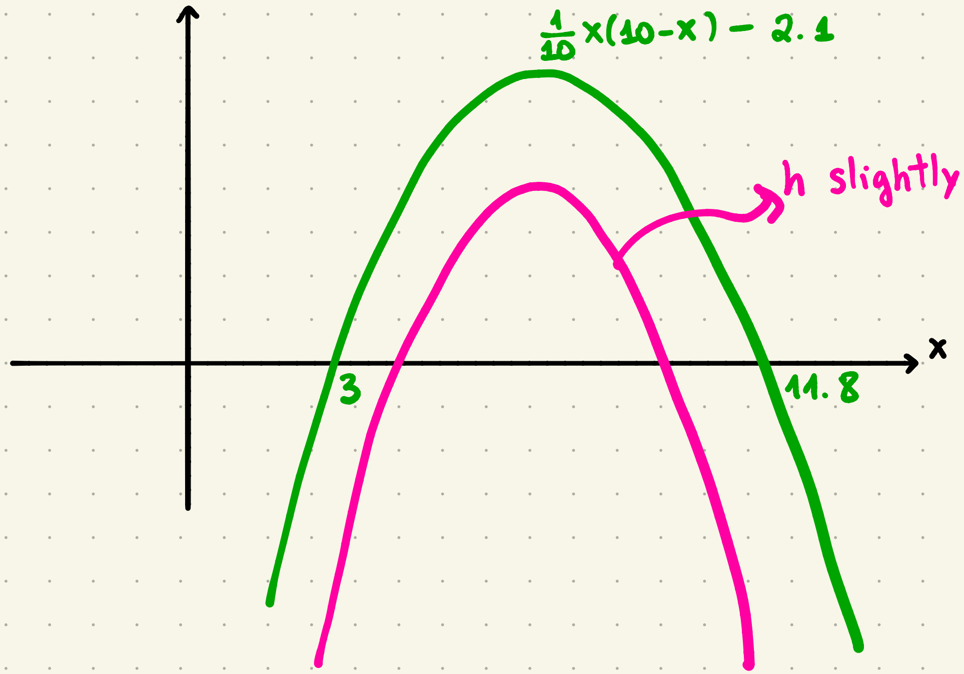
$$\Rightarrow \sqrt{100-40h}<4$$

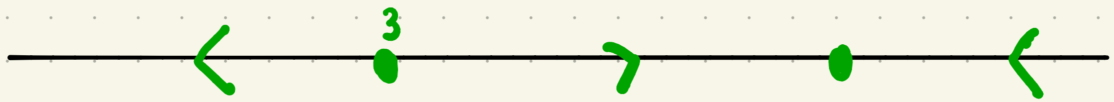
$$\Rightarrow 100-40h<16$$

$$\Rightarrow h>2.1$$

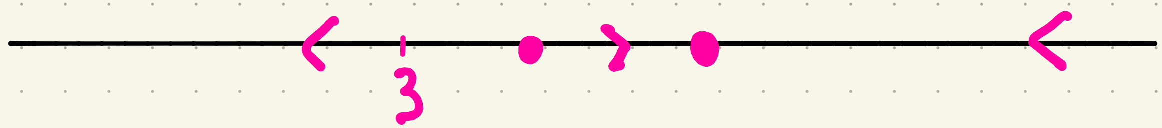
ANSWER: the solution reaches 0 when $h>2.1$.

Let's check the answer by graphing $\frac{1}{10}x(10-x)-h$ and the phase diagram of $\frac{dx}{dt}=\frac{1}{10}x(10-x)-h$.
(do an accurate plot by computer)





3 is a critical value



solution that starts at 3 decreases to $-\infty$

BIFURCATIONS

Consider the one-parameter family of autonomous equations

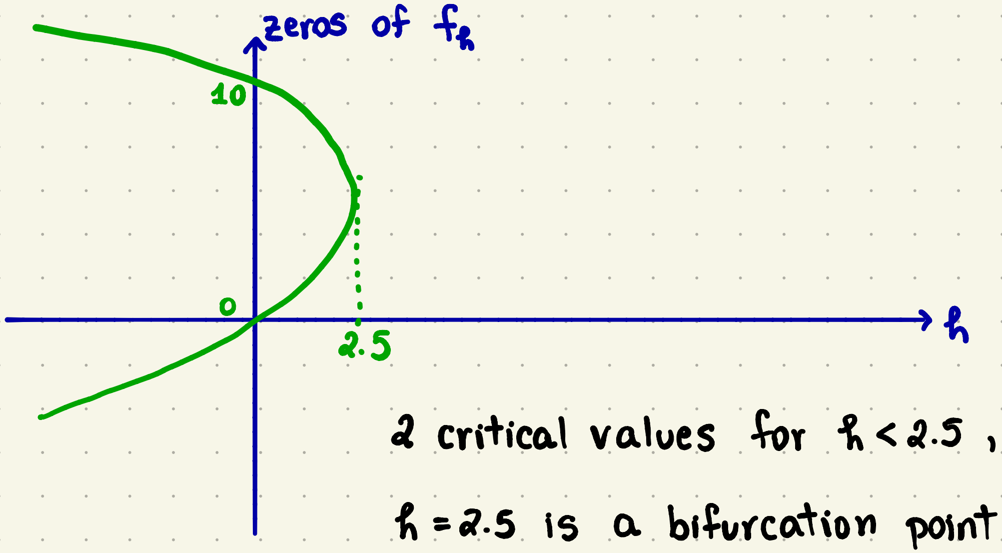
$$p' = f_h(p)$$

(for each number h, a different equation)

DEFINITION: a number h_* is said to be a **bifurcation point** (for the family $p'=f_h(p)$) if the number of critical values of $p'=f_h(p)$ for h near h_* is different from the number of critical values of $p'=f_{h_*}(p)$.

DEFINITION: the **bifurcation diagram** for the family $p'=f_h(p)$ is a graph of the critical values of $p'=f_h(p)$ (i.e. the zeros of against h). It may NOT be the graph of a function!

EXAMPLE: bifurcation diagram for $x' = \frac{1}{10}x(10-x) - h =: f_h(x)$



Given h, what are the zeros of f_h?

$$0 = \frac{1}{10}x(10-x) - h$$

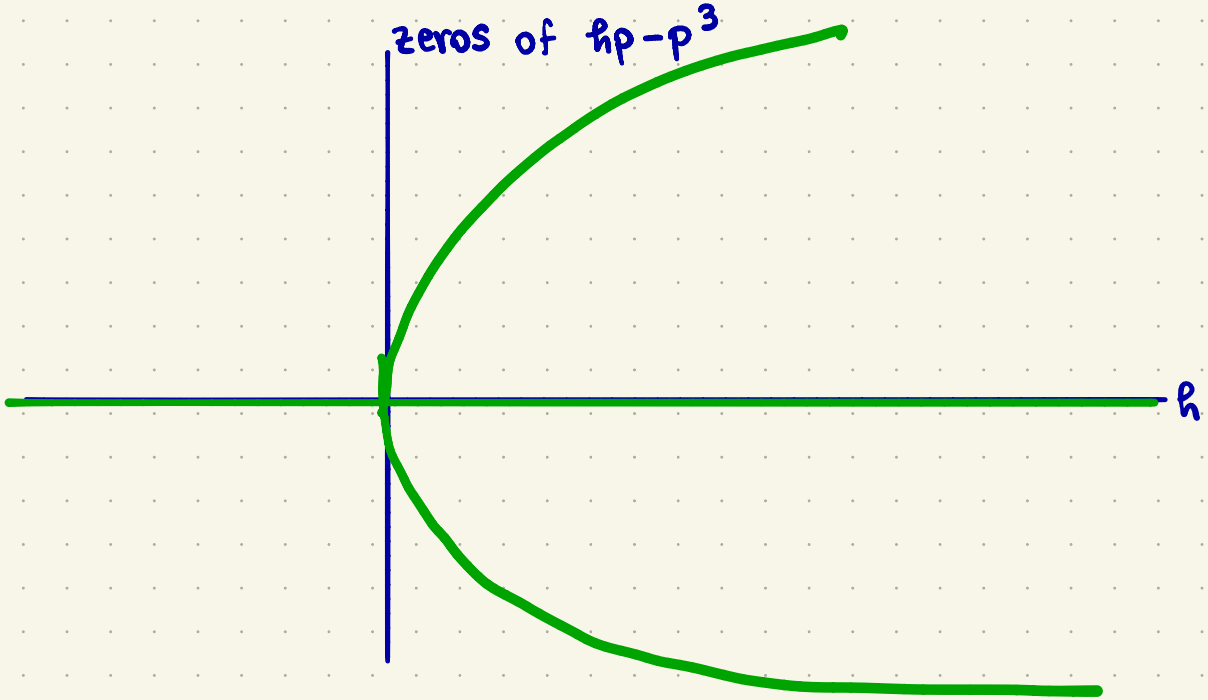
$$x = \frac{1}{2}\left(10 \pm \sqrt{100-40h}\right)$$

$$h = -\frac{1}{10}x^2 + x$$

2 critical values for $h<2.5$, 1 critical value for $h=2.5$, no critical value for $h>2.5$

$h=2.5$ is a bifurcation point

EXAMPLE: bifurcation diagram for $p' = hp - p^3$



Given h, which p make $hp - p^3 = 0$?

$p=0$ always works

if $p \neq 0$ and $hp - p^3 = 0$ then $p^2 = h$

1 critical value for $h \leq 0$, 3 critical values for $h > 0$

0 is a bifurcation point

Google 'bifurcation diagram' to see some cool pictures