2.3 ACCELERATION-VELOCITY MODELS

PROBLEM 1: A motorboat is moving at 40ft/s when its motor quits, and 10s later its speed is 20ft/s. Assuming the resistance is proportional to the square of the velocity, how far does the motorboat coast in the first minute after its motor quits?

Let $v(t)$ be the speed of the motorboat (in ft/s) t seconds after its motor stops.

GOAL: compute $\int_0^{60} v(t)\,dt$.

GIVEN: $v(0)=40$, $v(10)=20$, $v'(t)=-k\,v(t)^2$ for some $k>0$.

PLAN: solve $v'=-kv^2$ (there will be two unknowns, K and C)

then use $v(0)=40$ and $v(10)=20$ to solve for k and C

then compute $\int_0^{60} v(t)\,dt$

EXECUTION OF THE PLAN: $v'=-kv^2 \Rightarrow v^{-2}dv=-k\,dt \Rightarrow -v^{-1}=-kt+C \Rightarrow v=\dfrac{1}{kt+C}$ $C=-(\text{previous } C)$

$$v(0)=40 \Rightarrow 40=\frac{1}{C} \Rightarrow C=\frac{1}{40}$$

$$v(10)=20 \Rightarrow 20=\frac{1}{10k+\frac{1}{40}} \Longrightarrow 10k=\frac{1}{20}-\frac{1}{40}=\frac{1}{40} \Rightarrow k=\frac{1}{400}$$

$$\int_0^{60} v(t)\,dt=\int_0^{60}\frac{1}{\frac{t}{400}+\frac{1}{40}}\,dt=\ln\left(\frac{t}{400}+\frac{1}{40}\right)\cdot 400\Big|_0^{t=60}=400\ln 7\approx 778.36$$

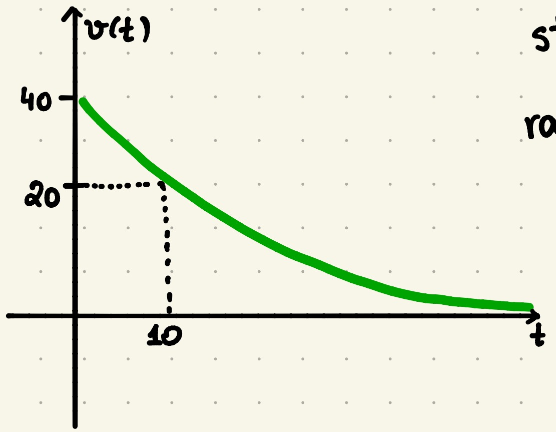
ANSWER: The motorboat coasted for 778.36 ft in the first minute after its motor stopped.

DOES THE ANSWER MAKE SENSE?

If there was no resistance, the motorboat would coast for $40\frac{ft}{s} \cdot 60s = 2400 ft$ in one minute, which is much more than the 778.36 ft we found, and that makes sense.

We found for the speed of the motorboat the formula

$$v(t) = \left(\frac{t}{400} + \frac{1}{40}\right)^{-1}$$



starts at 40 ft/s, drops to 20 ft/s in 10s, keeps dropping at ever slower rates, which makes sense

IS THE ASSUMPTION THAT RESISTANCE IS PROPORTIONAL TO THE SQUARE OF VELOCITY REASONABLE?

We don't know. This has to be found by experiment. The problem we just solved and problem 3 in HW 11 suggest an <u>experiment</u> to find how resistance depends on speed.

PROBLEM 2: A woman bails out of an airplane at 10,000 ft, falls freely for 20 s, then opens her parachute. How long will it take her to hit the ground? Assume air resistance proportional to the speed, with constant of proportionality 0.15/s without the parachute and 1.5/s with the parachute. Assume constant gravitational acceleration of $32\,ft/s^2$.

Let $v(t)$ be the donward speed in ft/s after t seconds of bailing out.

WANT: a time t_0 such that $\int_0^{t_0} v(t)\,dt = 10{,}000$.

GIVEN: $v(0)=0$, $\quad v'(t) = \begin{cases} -0.15\,v(t)+32, & t<20 \\ -1.5\,v(t)+32, & t \geq 20 \end{cases}$

PLAN: 1) solve $v'(t) = -0.15\,v(t)+32$, $v(0)=0$, compute $v(20)$.

2) solve $v'(t+20) = -1.5\,v(t+20)+32$ with $v(20)$ computed in the first step.

3) compute $\int_0^{t_0} v(t)\,dt$ and solve for the t_0 that makes this integral 10,000.

EXECUTING THE PLAN:

1) $v' + 0.15v = 32$, $v(0)=0$ LINEAR

Integrating factor $\exp\left(\int 0.15\,dt\right) = e^{0.15t}$

Multiply eq. by the int. factor, then integrate:

$$\left(e^{0.15t} v\right)' = 32\,e^{0.15t} \Rightarrow e^{0.15t} v = \frac{32}{0.15} e^{0.15t} + C \Rightarrow v = \frac{32}{0.15} + Ce^{-0.15t}$$

Use $v(0)=0$ to find C and get $C = -32/0.15$.

Compute $v(20) = \frac{32}{0.15} - \frac{32}{0.15} e^{-3} \approx 202.71$

2) $v' = -1.5v + 32$, $v(20) = 202.71$

Solve as in step 1 : $v(t) = \frac{32}{1.5} + e^{-1.5(t-20)}\left(202.71 - \frac{32}{1.5}\right)$ for $t \geqslant 20$

$$v(t) = \begin{cases} \frac{32}{0.15}\left(1 - e^{-0.15t}\right), & t < 20 \\ \frac{32}{1.5} + 181.38\, e^{-1.5(t-20)}, & t \geqslant 20 \end{cases}$$

3) Assume $t_0 > 20$. Then

$$\int_0^{t_0} v(t)\,dt = \int_0^{20} \frac{32}{0.15}\left(1 - e^{-0.15t}\right)dt + \int_{20}^{t_0} \frac{32}{1.5} + 181.38 \cdot e^{-1.5(t-20)}\,dt$$

$$= \frac{32}{0.15} \times 20 - \frac{32}{(0.15)^2}\left(1 - e^{-0.15 \times 20}\right) + \frac{32}{1.5}(t_0 - 20) + \frac{181.38}{1.5}\left(1 - e^{-1.5(t_0 - 20)}\right)$$

Solving (on a computer) we find $t_0 \approx 346$.

<u>ANSWER</u>: it takes approximately 5min 46s for the woman to hit the ground.

PROBLEM 3: A rocket of mass 10kg is launched upward with initial velocity 20 m/s from a platform that is 3m high. Assume air resistance proportional to the velocity with constant 1/s, and assume constant gravitational acceleration 9.8m/s².

A) Find the maximum height reached by the rocket.
B) How long does it take for it to reach the ground?

Let $v(t)$ be the upward velocity in m/s after t seconds of launching.

A) <u>GOAL</u>: compute $\int_0^{t_0} v(t)\,dt + 3$, where $v(t_0)=0$.

<u>GIVEN</u>: if $t<t_0$ (rocket moving upward) then $\begin{cases} mv'(t) = -v(t) - mg, \text{ where } m=10,\ g=9.8 \\ v(0)=20 \end{cases}$ if the units of measurement change, the equation also changes!

<u>PLAN</u>: 1) solve the IVP
2) solve for t_0 in $v(t_0)=0$
3) compute $3+\int_0^{t_0} v(t)\,dt =: h_{max}$ ← the symbol =: means 'the thing on the right is DEFINED to be equal to the thing on the left'

B) Let $w(t)$ be the DOWNWARD speed, in m/s. t seconds after reaching maximal height.

<u>GOAL</u>: find t_1 such that $\int_0^{t_1} w(t)\,dt = h_{max}$, compute t_0+t_1

<u>GIVEN</u>: $(IVP_2)\begin{cases} mw'(t) = -w(t) + mg \\ w(0) = 0 \end{cases}$

<u>PLAN</u>: 1) solve for w in (IVP_2)
2) solve for t_1 in $\int_0^{t_1} w(t)\,dt = h_{max}$ (we know h_{max} from the first part of the problem)

EXECUTING THE PLAN:

A) 1) $mv'(t) = -v(t) - mg \Rightarrow v'(t) + \frac{1}{m}v(t) = -g \Rightarrow v(t) = -mg + e^{-t/m}(v(0) + mg)$

Plugging $v(0) = 20$, $m = 10$, $g = 9.8$, we get $v(t) = -98 + e^{-t/10}(20 + 98) = 118e^{-t/10} - 98$

2) Solve for t_0 in $v(t_0) = 0$:

$$0 = 118e^{-t_0/10} - 98 \Rightarrow t_0 = -10 \cdot \ln\frac{98}{118} \approx 1.86$$

3) Integrate speed to find distance traveled:

$$3 + \int_0^{1.86} 118e^{-t/10} - 98\,dt = 1180(1 - e^{-0.186}) - 98 \times 1.86 \approx 21$$

ANSWER TO (A): the maximum height attained by the rocket is approximately 21m.

B) 1) $mw' = -w + mg \Rightarrow w(t) = mg + e^{-t/m}(w(0) - mg)$

Plug $w(0) = 0$, $m = 10$, $g = 9.8$: $w(t) = 98 - 98e^{-t/10}$

2) solve for t_1 in $\int_0^{t_1} 98 - 98e^{-t/10}\,dt = 21$

$$98t_1 - 980(1 - e^{-t_1/10}) = 21$$

$t_1 \approx 2.14$ ← the time from maximum height to hitting the ground is ≈ 2.14s

Total time (going up + falling) $\approx 1.86 + 2.14 = 4$

ANSWER TO (B): it takes the rocket approximately 4 seconds to hit the ground.