2.4 2.5 EULER'S METHOD

It's an algorithm that produces approximations to the solution of

$$y' = f(t,y), \quad y(t_0) = y_0.$$

INPUT: a function $f(t,y)$ (slope field)

a point (t_0, y_0) (initial condition)

a positive number h (step size)

OUTPUT: a sequence $y_0, y_1, y_2, y_3, \ldots$ $(y_k \approx y(kh))$

PROCEDURE: $t_{k+1} := t_k + h$

$$y_{k+1} := y_k + h \cdot f(t_k, y_k)$$

EXAMPLE: consider the IVP $\quad y' = y^2 + t^2, \quad y(0) = 1.$

Find an approximation to $y(0.2)$ using Euler's method with step size 0.1.

Let y_1 and y_2 be approximations to $y(0.1)$ and $y(0.2)$, respectively. Let $f(t,y) := y^2 + t^2$.

$$y_1 = 1 + 0.1\, f(0,1) = 1 + 0.1\,(0^2 + 1^2) = 1.1$$

$$y_2 = y_1 + 0.1\, f(0.1, y_1) = 1.1 + 0.1\left(0.1^2 + 1.1^2\right) = 1.1 + 0.122 = 1.222$$

EXAMPLE: Consider $y' = 2y$, $y(0) = 1$. Compute approximations to $y(\frac{1}{2})$ using step sizes 0.25 and 0.1, and compare the results with the true value of $y(\frac{1}{2})$.

Step size 0.25

$$y(0.25) \approx y_1 = 1 + 0.25 \times (2 \times 1) = 1.5$$

$$y(0.5) \approx y_2 = 1.5 + 0.25 \times (2 \times 1.5) = 2.25$$

Step size 0.1

$$y(0.1) \approx y_1 = 1 + 0.1 \times (2 \times 1) = 1.2$$

$$y(0.2) \approx y_2 = 1.2 + 0.1 \times (2 \times 1.2) = 1.44$$

$$y(0.3) \approx y_3 = 1.44 + 0.1 \times (2 \times 1.44) = 1.728$$

$$y(0.4) \approx y_4 = 1.728 + 0.1 \times (2 \times 1.728) = 2.0736$$

$$y(0.5) \approx y_5 = 2.0736 + 0.1 \times (2 \times 2.0736) = 2.48832$$

IMPROVED EULER'S METHOD:

INPUT: a function $f(t,y)$ (slope field)

a point (t_0, y_0) (initial condition)

a positive number h (step size)

OUTPUT: a sequence $y_0, y_1, y_2, y_3, \ldots$ $(y_k \approx y(kh))$

PROCEDURE:

$$t_{k+1} := t_k + h$$

$$u_{k+1} := y_k + h \cdot f(t_k, y_k) \qquad \text{(Euler approximation)}$$

$$y_{k+1} := y_k + h\,\frac{f(t_k, y_k) + f(t_{k+1}, u_{k+1})}{2} \qquad \text{(improved Euler approximation)}$$