3.1 SECOND-ORDER LINEAR EQUATIONS

DEFINITION: The order of a diff. eq. is the order of the highest derivative in the equation.

DEFINITION: A linear diff. eq. has the form

$$P_0(x)\,y^{(n)} + P_1(x)\,y^{(n-1)} + \cdots + P_n(x)\,y = F(x)$$

for some n. It has constant coefficients if $P_0(x), P_1(x), \ldots, P_n(x)$ are constant functions. It is homogeneous if $F(x) = 0$ for all x.

EXAMPLES:

$y'' + xy' + 3y = 0$	2nd order linear homogeneous
$y'' + xy' + 3y = x^2$	2nd order linear, not homogeneous
$y'' = y$	2nd order linear homogeneous constant coefficients
$yy' = x^2 + 2$	1st order nonlinear
$\sin(y'') - y' + 5y = 0$	2nd order nonlinear
$y''' + 3x^2 y' = 7x$	3rd order linear, not homogeneous

PRINCIPLE OF SUPERPOSITION: consider a linear homogeneous equation

$$P_0(x)y^{(n)} + P_1(x)y^{(n-1)} + \cdots + P_n(x)y = 0 \qquad (\star)$$

If $y_1(x)$ and $y_2(x)$ solve $(\star)$ then $Ay_1(x) + By_2(x)$ also does, for any constants A and B.

EXAMPLE: $y'' = -y$

$$\left.\begin{array}{l} y_1(x) = \sin x \text{ is a solution} \\ y_2(x) = \cos x \text{ is also a solution} \end{array}\right\} \Longrightarrow y(x) = A\sin x + B\cos x \text{ is a solution}$$

Are there more solutions? Does $A\sin x + B\cos x$ include all possible solutions? The answer comes from the Existence and Uniqueness Theorem.

THEOREM (EXISTENCE AND UNIQUENESS): the IVP

$$P_0(x)y^{(n)} + P_1(x)y^{(n-1)} + \cdots + P_n(x)y = F(x), \quad y(x_0) = y_0,\ y'(x_0) = y_1, \ldots, y^{(n-1)}(x_0) = y_{n-1}$$

has one and only one solution defined for x in the open interval I, provided $P_0(x), \ldots, P_n(x), F(x)$ are continuous in I and x_0 belongs to I.

COROLLARY: for a linear equation of order n, the general solution has n parameters.

2ND ORDER HOMOGENEOUS LINEAR EQUATIONS WITH CONSTANT COEFFICIENTS

$$ay'' + by' + cy = 0$$

By the existence and uniqueness theorem, if we can find two solutions $y_1(x)$, $y_2(x)$ that are NOT multiples of each other then the general solution is $C_1 y_1(x) + C_2 y_2(x)$.

GUESS $y(x) = e^{rx}$ for some r to be determined. Plug into the equation, solve for r

$$a(e^{rx})'' + b(e^{rx})' + c(e^{rx}) = 0$$

$$(ar^2 + br + c)e^{rx} = 0$$

$$ar^2 + br + c = 0$$ characteristic equation

If the characteristic equation has two distinct real roots then we have our two solutions.

EXAMPLE: Find the general solution of $2y''-y'-y=0$.

The equation is second-order linear homogeneous with constant coefficients.

It's characteristic equation is $2r^2-r-1=0$, whose zeros are $\frac{1\pm\sqrt{1-4\times2\times(-1)}}{4}=1$ or $-\frac{1}{2}$.

Since the roots are distinct, the solutions $y_1(t)=e^t$ and $y_2(t)=e^{-t/2}$ are LI.

Therefore the general solution is $\boxed{y(t)=Ae^t+Be^{-t/2}}$

EXAMPLE: Find the general solution of $6y''-7y'-20y=0$.

The equation is second-order linear homogeneous with constant coefficients.

It's characteristic equation is $6r^2-7r-20=0$, whose zeros are $\frac{7\pm\sqrt{49-4\times6\times(-20)}}{12}=\frac{7\pm23}{12}=\frac{5}{2}$ or $-\frac{4}{3}$

Since the roots are distinct, the solutions $y_1(t)=e^{5t/2}$ and $y_2(t)=e^{-4t/3}$ are LI.

Therefore the general solution is $\boxed{y(t)=Ae^{5t/2}+Be^{-4t/3}}$

WHAT IF THE CHARACTERISTIC EQUATION HAS REPEATED ROOTS?

$$y'' - 2y' + y = 0$$

Characteristic equation $\quad r^2 - 2r + 1 = 0$

$$(r-1)^2 = 0$$

$\Rightarrow y_1(x) = e^x$ is a solution. Need another solution NOT of the form Ce^x.

GUESS $y_2(x) = xe^x$.

$$(xe^x)'' - 2(xe^x)' + xe^x \stackrel{?}{=} 0$$

$$(xe^x + e^x)' - 2(xe^x + e^x) + xe^x = 0$$

$$xe^x + 2e^x - 2xe^x - 2e^x + xe^x = 0$$

works!

So the general solution is $\quad y(x) = C_1 e^x + C_2 x e^x$.

Does it always work? Yes! Try it with $y'' - 2ay' + a^2 y = 0$.

Characteristic equation: $r^2 - 2ar + a^2 = 0 \iff (r-a)^2 = 0$.

$\Rightarrow e^{ax}$ is a solution of $y'' - 2ay' + a^2 y = 0$.

Try xe^{ax}:

$$(xe^{ax})'' - 2a(xe^{ax})' + a^2 xe^{ax} \stackrel{?}{=} 0$$

$$(0 \cdot e^{ax} + 2 \cdot 1 \cdot ae^{ax} + x \cdot a^2 e^{ax}) - 2ae^{ax} - 2a^2 xe^{ax} + a^2 xe^{ax} \stackrel{?}{=} 0 \quad \checkmark$$

WHAT IF THE CHARACTERISTIC EQUATION HAS COMPLEX ROOTS ?

$y'' + y = 0$

Char. eq. $r^2 = -1$.

Complex solutions e^{ix} and e^{-ix}. How to get real-valued solutions?

KEY OBSERVATION: if a complex-valued function solves a homogeneous linear equation then its real and imaginary parts also do.