3.2 GENERAL SOLUTIONS OF LINEAR EQUATIONS

DEFINITION: We say that a set $\{y_1, \dots, y_n\}$ of functions is linearly independent (LI) if

$$c_1 y_1 + c_2 y_2 + \cdots + c_n y_n = 0 \quad \text{implies} \quad c_1 = c_2 = \cdots = c_n = 0.$$

We say that $\{y_1, \dots, y_n\}$ is linearly dependent (LD) if one of the functions can be written as a linear combination of the remaining functions.

THINK: the functions are LI when there is no redundancy in the general solution $c_1 y_1 + \cdots + c_n y_n$

FACT: to solve a linear homogeneous equation of order n,

$$y^{(n)} + P_1(x)\, y^{(n-1)} + \cdots + P_n(x)\, y = 0$$

one needs to find n LI solutions $y_1, \dots, y_n$ and take linear combinations. In order words, the general solution is $c_1 y_1 + \cdots + c_n y_n$.

↪ with less than n LI solutions, it is impossible to generate all solutions

↪ if the linear eq. has order n, it is impossible to find $n+1$ LI solutions.

EXAMPLE: solve the IVP $y^{(3)} + 9y' = 0, \quad y(0) = 3, \ y'(0) = -1, \ y''(0) = 2 \quad \star$

given LI solutions $y_1 = 1, \ y_2 = \cos(3x), \ y_3 = \sin(3x)$.

General solution $y = A + B\cos(3x) + C\sin(3x) \qquad \star\star$

$$y' = -3B\sin(3x) + 3C\cos(3x)$$

$$y'' = -9B\cos(3x) - 9C\sin(3x)$$

From $\star$, we get

$$3 = A + B$$

$$-1 = +3C$$

$$2 = -9B$$

whence $C = -\frac{1}{3}$, $B = -\frac{2}{9}$, $A = \frac{29}{9}$. Going back to $\star\star$, we find $y = \frac{29}{9} - \frac{2}{9}\cos(3x) - \frac{1}{3}\sin(3x)$.

HOW DO WE KNOW IF A GIVEN SET OF FUNCTIONS IS LI?

THEOREM: let $y_1, y_2, \dots, y_n$ be solutions of the linear homogeneous equation

$$y^{(n)} + P_1(x)\, y^{(n-1)} + \cdots + P_n(x)\, y = 0.$$

Then $\{y_1, \dots, y_n\}$ is LI if and only if the following determinant is $\neq 0$ for some x.

$$W(x) := \begin{vmatrix} f_1(x) & f_2(x) & \cdots & f_n(x) \\ f_1'(x) & f_2'(x) & \cdots & f_n'(x) \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(n-1)}(x) & f_2^{(n-1)}(x) & \cdots & f_n^{(n-1)}(x) \end{vmatrix}.$$

EXAMPLE: check that $\{1, \cos(3x), \sin(3x)\}$ is LI.

Need to check $W(x) \neq 0$ for some x, where

$$W(x) = \begin{vmatrix} 1 & \cos(3x) & \sin(3x) \\ 0 & -3\sin(3x) & 3\cos(3x) \\ 0 & -9\cos(3x) & -9\sin(3x) \end{vmatrix} \qquad W(0) = \begin{vmatrix} 1 & 1 & 0 \\ 0 & 0 & 3 \\ 0 & -9 & 0 \end{vmatrix} = 27 \neq 0 \quad \text{done}$$

EXAMPLE: check that $\{e^{ax}, e^{bx}, e^{cx}\}$ is LI if a, b and c are distinct.

Need to check $W(x) \neq 0$ for some x, where

$$W(x) = \begin{vmatrix} e^{ax} & e^{bx} & e^{cx} \\ a e^{ax} & b e^{bx} & c e^{cx} \\ a^2 e^{ax} & b^2 e^{bx} & c^2 e^{cx} \end{vmatrix}$$

$$W(0) = \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}$$

This is a Vandermonde determinant (look it up).

Is it $\neq 0$? We can compute the 3×3 determinant by brute force:

$$W(0) = bc^2 - cb^2 - ac^2 + ca^2 + ab^2 - ba^2$$

$$= a^2(c-b) + b^2(a-c) + c^2(b-a)$$ ⇜ not clear if $\neq 0$

$$= a^2(c-b) + b^2(a-b+b-c) + c^2(b-a)$$

$$= (a-b)(b-c)(b+c) + (b-c)(b-a)(b+a)$$

$$= (a-b)(b-c)(c-a)$$

Clearly $\neq 0$ if a, b and c are distinct ☺

WHAT ABOUT NON-HOMOGENEOUS LINEAR EQUATIONS?

Consider the linear equation

$$y^{(n)} + P_1(x)\, y^{(n-1)} + \cdots + P_n(x) y = F(x). \qquad \text{(NH)}$$

Its associated homogeneous equation is, by definition,

$$y^{(n)} + P_1(x)\, y^{(n-1)} + \cdots + P_n(x) y = 0. \qquad \text{(H)}$$

FACT: the general solution of (NH) has the form

$$y(x) = y_p(x) + C_1 y_1(x) + \cdots + C_n y_n(x),$$

where $y_p(x)$ can be ANY solution of (NH) (often called a particular solution)

and $y_1, y_2, \dots, y_n$ are LI solutions of (H).