WHAT ABOUT NON-HOMOGENEOUS LINEAR EQUATIONS?

Consider the linear equation

$$y^{(n)} + P_1(x)\, y^{(n-1)} + \cdots + P_n(x)\, y = F(x). \qquad \text{(NH)}$$

Its associated homogeneous equation is, by definition,

$$y^{(n)} + P_1(x)\, y^{(n-1)} + \cdots + P_n(x)\, y = 0. \qquad \text{(H)}$$

FACT: the general solution of (NH) has the form

$$y(x) = y_p(x) + C_1 y_1(x) + \cdots + C_n y_n(x),$$

where $y_p(x)$ can be ANY solution of (NH) (often called a particular solution)

and $y_1, y_2, \ldots, y_n$ are LI solutions of (H).

GIVEN A SOLUTION, HOW TO FIND A LI SOLUTION?

One of the methods is called reduction of order. Consider

$$y'' + p(x)y' + q(x)y = 0 \qquad \text{(2nd order linear homogeneous)}$$

Let y_1 be a solution. How to find an independent solution y_2?

GUESS: $y_2(x) = v(x)y_1(x)$ for some $v(x)$ to be determined. Plug y_2 into the equation and try to solve for $v(x)$.

$$(vy_1)'' + p(x)(vy_1)' + q(x)vy_1 = 0$$

$$(v''y_1 + 2v'y_1' + vy_1'') + p(x)(v'y_1 + vy_1') + q(x)vy_1 = 0$$

$$v(\underbrace{y_1'' + p(x)y_1' + q(x)y_1}_{=0 \text{ because } y_1 \text{ is a solution}}) + y_1 v'' + (2y_1' + p(x)y_1)v' = 0$$

$$y_1 v'' + (2y_1' + p(x)y_1)v' = 0 \qquad \text{separable equation for } v'$$

EXAMPLE (CONSTANT COEFFICIENTS) Find two LI solutions to $y'' - 2ay' + a^2y = 0$

We know that $y_1 = e^{ax}$ and $y_2 = xe^{ax}$ work, and we know that y_1 can be found via the characteristic equation.

Plug $v(x)e^{ax}$ and try to solve for $v(x)$.

$$(v''e^{ax} + \cancel{2av'e^{ax}} + \cancel{a^2ve^{ax}}) - 2a(\cancel{v'e^{ax}} + \cancel{ave^{ax}}) + \cancel{a^2ve^{ax}} = 0$$

$$v'' = 0 \implies v(x) = Ax + B$$

EXAMPLE: Find the general solution of $x^2y''+xy'-y=0$, given that $y_1(x)=x$ is a solution.

Check that $y_1(x)=x$ is a solution: $x^2\cdot 0 + x\cdot 1 - x = 0$

Guess a solution $xv(x)$, plug into the equation and try to solve for v.

$$x^2(2v'+xv'')+x(\cancel{v}+xv')-x\cancel{v}=0$$

$$x^3v''=-3x^2v'$$

$$\frac{v''}{v'}=\frac{-3}{x}$$

$$\ln v' = -3\ln x + C$$

$$v' = e^C x^{-3}$$

$$v=\frac{e^Cx^{-2}}{-2}+D$$

Choosing $C=D=0$, we get the solution $-\frac{1}{2x}$. Check: $x^2\left(-\frac{1}{2x}\right)''+x\left(-\frac{1}{2x}\right)'-\left(-\frac{1}{2x}\right)\overset{?}{=}0$

$$x^2(-x^{-3})+\frac{1}{2}x^{-1}+\frac{1}{2x}\overset{\checkmark}{=}0$$

ANSWER: $y=Ax+\frac{B}{x}$.

EXAMPLE: Consider the linear homogeneous equation

$$(1-x^2)y'' - 2xy' + 2y = 0. \qquad \circledast$$

Given that $y_1(x) = x$ is a solution, find a linearly independent solution.

Guess $y_2(x) = x v(x)$ for a $v(x)$ that has to be determined.

$$(1-x^2)(xv)'' - 2x(xv)' + 2xv = 0$$

$$(1-x^2)(2v' + xv'') - 2x(\cancel{v} + xv') + 2\cancel{x}\cancel{v} = 0$$

$$(1-x^2)xv'' = v'(2x^2 - 2(1-x^2))$$

$$\frac{v''}{v'} = \frac{2x}{1-x^2} - \frac{2}{x}$$

$$\ln v' = -\ln(1-x^2) - 2\ln x + C$$

$$v' = \frac{e^C}{(1-x^2)x^2} = e^C\left(\frac{1}{x^2} + \frac{1}{1-x^2}\right) = e^C\left(\frac{1}{x^2} + \frac{1}{2(1+x)} + \frac{1}{2(1-x)}\right)$$

$$v = e^C\left(-x^{-1} + \frac{1}{2}\ln\frac{1+x}{1-x}\right)$$

To find a solution of $\circledast$ that is independent of x, choose a value of C and multiply c by x. For example, $\boxed{y_2 = -1 + \frac{x}{2}\ln\frac{1+x}{1-x}}$.

3.3 HOMOGENEOUS EQUATIONS WITH CONSTANT COEFFICIENTS

SUMMARY: consider $y'''' + c_1 y''' + c_2 y'' + c_3 y' + c_4 y = 0$ ⊛

It's characteristic equation is $\underbrace{r^4 + c_1 r^3 + c_2 r^2 + c_3 r + c_4}_{p(r)} = 0$

The solution of ⊛ depends on the factorization of $p(r)$.

$p(r)$	solution
$(r-a)(r-b)(r-c)(r-d)$	$Ae^{ax} + Be^{bx} + Ce^{cx} + De^{dx}$
$(r-a)^2(r-b)(r-c)$	$Ae^{ax} + Bxe^{ax} + Ce^{bx} + De^{cx}$
$(r-a)^3(r-b)$	$(A + Bx + Cx^2)e^{ax} + De^{bx}$
$(r-a)(r-b)(r-(c+id))(r-(c-id))$	$Ae^{ax} + Be^{bx} + Ce^{cx}\cos(dx) + De^{cx}\sin(dx)$
$(r-a)^2(r-(c+id))(r-(c-id))$	$(A+Bx)e^{ax} + e^{cx}(C\cdot\cos(dx) + D\sin(dx))$
$(r-(a+ib))(r-(a-ib))(r-(c+id))(r-(c-id))$	$e^{ax}(A\cos(bx) + B\sin(bx)) + e^{cx}(C\cdot\cos(dx) + D\cdot\sin(dx))$
$(r-(a+ib))^2(r-(a-ib))^2$	$(A+Bx)e^{ax}(C\cos(bx) + D\sin(bx))$
$(r-(a+ib))(r-(a-ib))(r-c)^2$	$(A\cos(bx) + B\sin(bx))e^{ax} + (C+Dx)e^{cx}$

EXAMPLE: $4y''+4y'+2y=0$

Characteristic equation: $4r^2+4r+2=0$

Roots: $r=\dfrac{-4\pm\sqrt{16-4\times 4\times 2}}{8}=\dfrac{-4\pm\sqrt{-16}}{8}=\dfrac{-1\pm i}{2}$

$\Rightarrow$ there are two LI solutions $y_1=e^{\frac{-1+i}{2}x}$ and $y_2=e^{\frac{-1-i}{2}x}$, but they take complex values.

How do we get a real-valued solution for them?

FACT: BECAUSE THE EQUATION IS LINEAR AND ITS COEFFICIENTS ARE REAL NUMBERS, the real and imaginary parts of y_1 and y_2 are solutions.

$$y_1(x)=e^{-x/2}e^{ix/2}=e^{-x/2}\left(\cos\left(\tfrac{x}{2}\right)+i\sin\left(\tfrac{x}{2}\right)\right)$$

$\Rightarrow$ $e^{-x/2}\cos\frac{x}{2}$ and $e^{-x/2}\sin\frac{x}{2}$ are solutions to $4y''+4y'+2=0$.

EXAMPLE: $y''-2y'+5y=0$

Characteristic equation: $r^2-2r+5=0$

Roots: $r=\dfrac{2\pm\sqrt{4-20}}{2}=1\pm 2i$

General solution: $y=Ae^x\cos(2x)+Be^x\sin(2x)$.