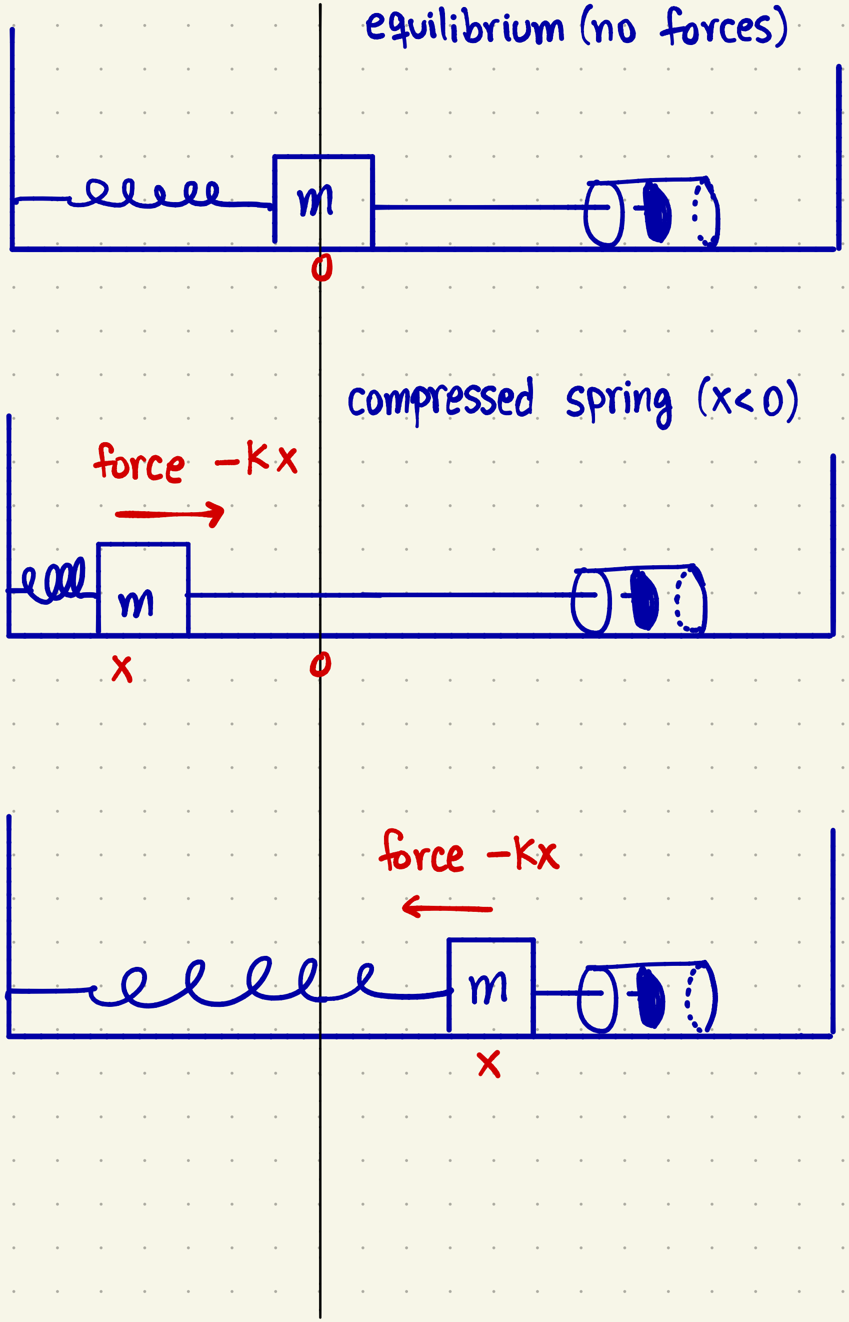
3.4 MECHANICAL VIBRATIONS



Consider a block of mass m attached to a spring and to a dashpot. When displaced from equilibrium a distance d, restorative force of magnitude kd.

When the block moves at speed v, the dashpot opposes the motion with a force of magnitude cv.

The displacement $x(t)$ at time t satisfies

$$m x''(t) + c x'(t) + k x(t) = 0$$

(2nd order, linear, homogeneous with constant coefficients)

GOAL FOR TODAY: classify the possible behaviors of the block and compute some quantities of the motion (period and amplitude of the oscillations).

FIRST CASE: UNDAMPED MOTION $(c=0)$

$$m x''(t) + K x(t) = 0$$

Characteristic equation $\quad m r^2 + K r = 0$

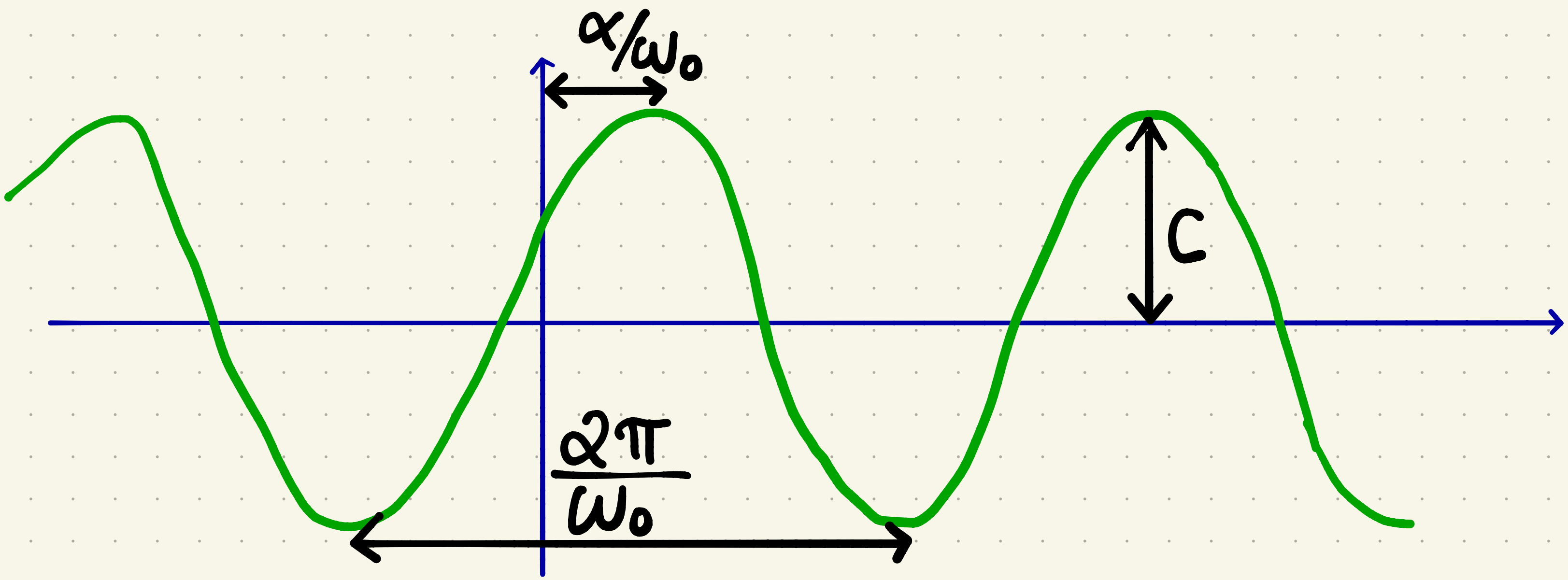
Roots $r = \pm \sqrt{\frac{K}{m}}\, i$.

$\Rightarrow$ General solution $\quad x(t) = A \cos\left(\sqrt{\frac{K}{m}}\, t\right) + B \sin\left(\sqrt{\frac{K}{m}}\, t\right)$.

Let $\omega_0 := \sqrt{\frac{K}{m}}$ and call it the frequency. Looking at the graph of some solutions and keeping in mind that the equation models the block on a spring, we may conjecture the following

PROPOSITION: For any A, B and ω_0, there are numbers $C > 0$ and α such that

$$A \cos(\omega_0 t) + B \sin(\omega_0 t) = C \cdot \cos(\omega_0 t - \alpha)$$



In fact, $C^2 = A^2 + B^2$ and $\tan \alpha = A/B$.

PROOF:

$$C\cdot\cos(\omega_0 t-\alpha)=C\cdot\cos(\omega_0 t)\cos\alpha+C\cdot\sin(\omega_0 t)\sin\alpha$$

$$=A\cos(\omega_0 t)+B\cdot\sin(\omega_0 t)$$

$$\Rightarrow A^2+B^2=(C\cdot\cos\alpha)^2+(C\cdot\sin\alpha)^2=C^2 \quad \text{and} \quad \cos\alpha=A/C\,.$$

PROBLEM: express C and α in terms of $x(0)$ and $x'(0)$.

$$x(t)=C\cdot\cos(\omega_0 t-\alpha)\Rightarrow x(0)=C\cdot\cos(-\alpha)=C\cdot\cos\alpha \qquad (\mathrm{I})$$

$$x'(t)=-\omega_0 C\cdot\sin(\omega_0 t-\alpha)\Rightarrow x'(0)=-\omega_0 C\cdot\sin(-\alpha)=\omega_0 C\cdot\sin\alpha \qquad (\mathrm{II})$$

Dividing (II) by (I), we get $\frac{x(0)}{x'(0)}=\omega_0\cdot\tan\alpha \Rightarrow \alpha=\operatorname{atan}\left(\frac{x(0)}{\omega_0 x'(0)}\right)$

Besides, $\omega_0^2 x(0)^2+x'(0)^2=\omega_0^2 C^2 \Rightarrow C=\sqrt{x(0)^2+\frac{x'(0)^2}{\omega_0^2}}$

REMARK: the period ω_0 does not depend on the initial condition!

SECOND CASE: DAMPED MOTION ($c > 0$)

$$mx''(t) + cx'(t) + kx(t) = 0$$

Characteristic equation: $mr^2 + cr + k = 0$

Roots: $r = \dfrac{-c \pm \sqrt{c^2 - 4mk}}{2m}$

3 possibilites depending on the sign of $c^2 - 4mk$.

1st possibility: overdamped ($c^2 > 4mk$)

$x(t) = Ae^{r_1 t} + Be^{r_2 t}$, where r_1 and r_2 are the roots of the characteristic equation.

2nd possibility: critically damped ($c^2 = 4mk$)

Double root $\frac{-c}{2m} =: -p$

$x(t) = (A + Bt)e^{-pt}$

3rd possibility: underdamped ($c^2 < 4mk$)

Roots $\frac{-c}{2m} \pm \sqrt{\frac{c^2}{4m^2} - \frac{k}{m}} = -p \pm i\sqrt{\omega_0^2 - p^2}$

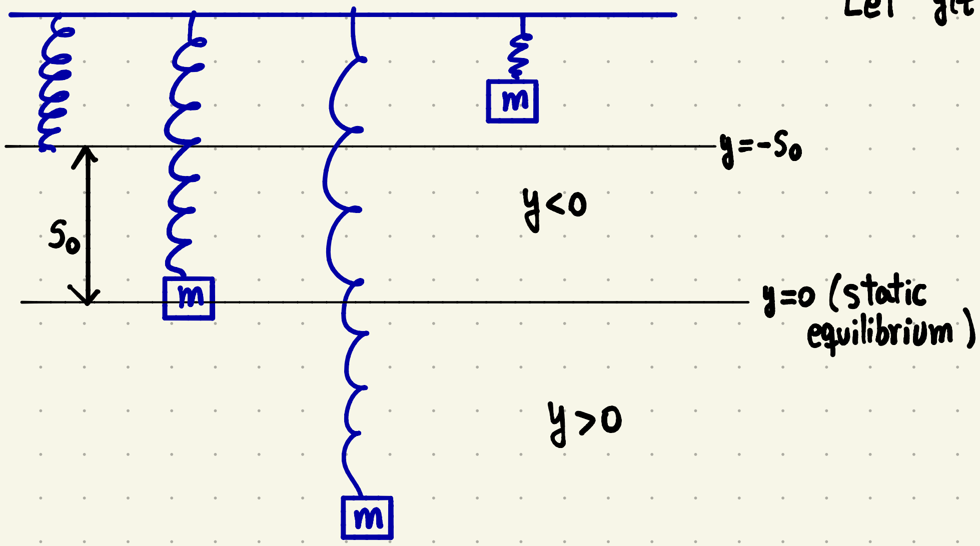
Let $\omega_1 := \sqrt{\omega_0^2 - p^2}$ and call it circular frequency.

$x(t) = e^{-pt}(A\cos\omega_1 t + B\sin\omega_1 t)$

$= C \cdot e^{-pt}\cos(\omega_1 t - \alpha)$

Then $T_1 = \frac{2\pi}{\omega_1}$ is called pseudo-period.

EXAMPLE: Consider a block of mass m that is attached to a spring suspended vertically. It stretches the spring a distance s_0. Let k be the spring constant and c be the friction constant (assume friction proportional to velocity). Find an equation for the motion of the block.



Let $y(t)$ be the distance from static equilibrium, increasing downwards.

At static equilibrium, $mg = ks_0$, so $k = \frac{mg}{s_0}$.

In general,

$$my''(t) = mg - k(y(t)+s_0) - cy'(t)$$

$$\Rightarrow \boxed{my'' + cy' + ky = 0}$$

The exact same equation as for the horizontal spring.