3.5 NONHOMOGENEOUS EQUATIONS, UNDETERMINED COEFFICIENTS AND VARIATION OF PARAMETERS

1) SOLUTION = PARTICULAR + HOMOGENEOUS

Consider the linear equation $y^{(3)} + p(t)y'' + q(t)y' = F(t)$. (NH)

If $F \neq 0$ then the superposition principle does NOT hold.

$y' = 1$ has solutions x and $x+1$, but $2x+1$ is not a solution

$y'' = y + t$ has solutions $-t$ and $e^t - t$, but $e^t - 2t$ is not a solution. Neither is $2(e^t - t)$.

However if y and y_p are solutions of (NH) then $y - y_p$ is a solution of

$$y^{(3)} + p(t)y'' + q(t)y' = 0 \quad \text{(H)},$$

named the homogeneous equation associated to (NH).

And we know all solutions of (H). Therefore

THEOREM: The general solution of (NH) is

$$y = y_p + Ay_1 + By_2 + Cy_3,$$

where y_p is ANY solution of (NH) (called a particular solution) and $Ay_1 + By_2 + Cy_3$ is the general solution of (H).

REMARK: if you replace y_p by a different solution, say $\tilde{y}_p$, the formula still yields the same functions, because $y_p - \tilde{y}_p = ay_1 + by_2 + cy_3$ for some choice of a, b, c.

EXAMPLES: A) $y' = 1$

General solution $y = t + C = (t+1) + C = (t-2) + C = \cdots$

B) $y'' = y + t$

General solution $y = t + Ae^{t} + Be^{-t}$.

2) THE METHOD OF UNDETERMINED COEFFICIENTS.

It's a method that works (sometimes) to find particular solutions of linear eqs with constant coefficients.

EXAMPLE: $y'' + y = \sin x$

First, we look for a particular solution. GUESS $y = a x \cos x + b x \sin x$.

Why this guess? Because $\cos x$ and $\sin x$ will give $y'' + y = 0$. Multiplying by x (as in the case of a homogeneous equation whose characteristic equation has a double root) there is a chance that the derivatives will cancel the factors of x and only $\sin x$ survives in the end.

$$y = a x \cos x + b x \sin x$$

$$y' = a(x(-\sin x) + 1 \cdot \cos x) + b(x \cos x + 1 \cdot \sin x) = b x \cos x - a x \sin x + a \cos x + b \sin x$$

$$y'' = b(-x \sin x + \cos x) - a(x \cos x + \sin x) - a \sin x + b \cos x$$

Solve for a and b in $y'' + y = \sin x$

$$\begin{aligned} \sin x = y'' + y &= x \sin x(-b + b) \\ &+ x \cos x(-a + a) \\ &+ \sin x\,(-a - a) \\ &+ \cos x\,(b + b) = -2a \sin x + 2b \cos x \Rightarrow a = -\tfrac{1}{2}, \; b = 0 \end{aligned}$$

We found the solution $y = -\frac{1}{2} x \cos x$.

Check! $y' = -\frac{1}{2}(-x \sin x + \cos x)$, $y'' = \frac{1}{2}(x \cos x + \sin x) + \frac{1}{2} \sin x$, $y'' + y = \frac{1}{2} \sin x + \frac{1}{2} \sin x = \sin x$ ☺

EXAMPLE: $y''-y'-2y=4x^2$, $y(0)=0$, $y'(0)=1$. $-2x^2+2x-3+\frac{2}{3}e^{2x}+\frac{7}{3}e^{-x}$

GUESS $\quad y=ax^2+bx+c$

$$y'=2ax+b$$

$$y''=2a$$

$$4x^2=y''-y'-2y = x^2(-2a)+x(-2a-2b)+(2a-b-2c)$$

$$\Rightarrow a=-2$$

$$4-2b=0 \Rightarrow b=2$$

$$-4-2-2c=0 \Rightarrow c=-3$$

A particular solution is $y=-2x^2+2x-3$.

Check! $(-2x^2-2x-3)''-(-2x^2+2x-3)'-2(-2x^2+2x-3) = -4-(-\cancel{4}x+2)+(4x^2-\cancel{4}x+6) = 4x^2$ ☺

EXAMPLE: $y''' - y'' - y' + y = 2e^{-t} + 3$ (Given: $r^3 - r^2 - r + 1 = (r-1)^2(r+1)$)

GUESS : $y = ate^{-t} + b$ (a guess with e^{-t} will not work because e^{-t} solves the homogeneous eq)

$$y' = at(-e^{-t}) + ae^{-t} = -ate^{-t} + ae^{-t}$$

$$y'' = -at(-e^{-t}) - ae^{-t} - ae^{-t} = ate^{-t} - 2ae^{-t}$$

$$y''' = at(-e^{-t}) + ae^{-t} + 2ae^{-t} = -ate^{-t} + 3ae^{-t}$$

$$\begin{aligned} 2e^{-t} + 3 = y''' - y'' - y' - y &= -ate^{-t} + 3ae^{-t} \\ &\quad - (ate^{-t} - 2ae^{-t}) \\ &\quad - (-ate^{-t} + ae^{-t}) \\ &\quad + (ate^{-t} + b) \\ &= 4ae^{-t} + b \end{aligned}$$

$$\Rightarrow \quad 4a = 2,\ b = 3 \quad \Rightarrow a = \frac{1}{2},\ b = 3$$

We found a particular solution $\boxed{y_p = \frac{1}{2}te^{-t} + 3}$.

3) THE METHOD OF VARIATION OF PARAMETERS

GOAL: find a particular solution to

$$y'' + p(x)y' + q(x)y = F(x). \qquad \text{(NH)}$$

METHOD: 1) Find two LI solutions y_1 and y_2 to

$$y'' + p(x)y' + q(x)y = 0, \qquad \text{(H)}$$

2) Find two functions v_1 and v_2 that satisfy

$$\begin{cases} v_1' y_1 + v_2' y_2 = 0 \\ v_1' y_1' + v_2' y_2' = F \end{cases}$$

3) Then $y_p := v_1 y_1 + v_2 y_2$ is a particular solution of (NH).

EXAMPLE: $y'' + y = \frac{1}{\sin x}$

1) Find two LI solutions of $y''+y=0$. We can take $\sin x$ and $\cos x$.

2) Find u and v that satisfy

$$v_1' \sin x + v_2' \cos x = 0 \qquad \text{(I)}$$

$$v_1' \cos x - v_2' \sin x = \frac{1}{\sin x} \qquad \text{(II)}$$

From (I), $v_1' = -v_2' \frac{\cos x}{\sin x}$. Plugging into (II),

$$v_2'\left(-\frac{\cos^2 x}{\sin x} - \sin x\right) = \frac{1}{\sin x} \Rightarrow v_2' = -1, \; v_1' = \frac{\cos x}{\sin x}.$$

Can take $v_2 = -x$ and $v_1 = \ln|\sin x|$.

3) A particular solution is

$$\boxed{y_p = \ln|\sin x| \sin x - x \cdot \cos x}$$

Test that it is a solution: $y_p' = \cancel{\cos} x + \ln|\sin x| \cos x - \cancel{\cos} x + x \sin x$

$$y_p'' = \frac{\cos^2 x}{\sin x} - \ln|\sin x| \sin x + x\cos x + \sin x$$

$$y_p'' + y_p = \frac{\cos^2 x}{\sin x} + \frac{\sin^2 x}{\sin x} = \frac{1}{\sin x}$$

EXAMPLE: $y'' - y' - 2y = 4x^2$

1) Find LI solutions for the homogeneous equation

$$y'' - y' - 2y = 0$$

Characteristic equation $\underbrace{r^2 - r - 2}_{(r+1)(r-2)} = 0$ Can take $y_1 = e^{-x}$, $y_2 = e^{2x}$

$$(e^{2x})'' - (e^{2x})' - 2e^{2x} = e^{2x}(4-2-2) = 0$$ ☺

2) Find v_1 and v_2 such that

$$v_1' e^{-x} + v_2' e^{2x} = 0 \quad \text{(I)}$$

$$-v_1' e^{-x} + 2 v_2' e^{2x} = 4x^2 \quad \text{(II)}$$

(I)+(II): $3v_2' e^{2x} = 4x^2 \Rightarrow v_2' = \frac{4}{3} x^2 e^{-2x}$

2(I)−(II): $3v_1' e^{-x} = -4x^2 \Rightarrow v_1' = -\frac{4}{3} x^2 e^{x}$

$$v_2 = \int \frac{4}{3} x^2 e^{-2x} dx$$

$$= -\frac{2}{3} x^2 e^{-2x} - \int -\frac{4}{3} x e^{-2x} dx$$

$$= -\frac{2}{3} x^2 e^{-2x} - \frac{2}{3} x e^{-2x} - \int -\frac{2}{3} e^{-2x} dx$$

$$= \frac{2}{3} e^{-2x} \left(-x^2 - x - \frac{1}{2}\right)$$

$v_1 = \int -\frac{4}{3} x^2 e^x dx = -\frac{4}{3} e^x Q(x)$ for some polynomial $Q(x)$, that should satisfy

$$Q(x) + Q'(x) = x^2$$

Let $Q(x) = ax^2 + bx + c$ $\quad a = 1, b = -2, c = 2$

$Q'(x) = 2ax + b$

$$v_1 = -\frac{4}{3} e^x (x^2 - 2x + 2)$$

3) A particular solution is $y_p = v_1 y_1 + v_2 y_2$.

$$y_p = -\frac{4}{3}(x^2 - 2x + 2) + \frac{2}{3}\left(-x^2 - x - \frac{1}{2}\right)$$

$$= -2x^2 + 2x - 3$$

4) Check wheter it works!

$$(-2x^2 + 2x - 3)'' - (-2x^2 + 2x - 3)' - 2(-2x^2 + 2x - 3) \overset{?}{=} 0$$

$$-4 - (-4x + 2) + 4x^2 - 4x + 6 \overset{?}{=} 0$$ TRUE! ☺

REMARK: we have solved the same example by undetermined coefficients.