3.6 FORCED OSCILLATIONS AND RESONANCE

On Section 3.4, we analyzed the eq.

$$mx'' + cx' + kx = 0$$

and found the following types of solutions:

1) $c=0$ (undamped) $x(t) = C\cdot\cos(\omega_0 t - \alpha)$, where $\omega_0 = \sqrt{\frac{k}{m}}$

2) $c^2 > 4mk$ (overdamped) $x(t) = Ae^{-r_1 t} + Be^{-r_2 t}$

3) $c^2 = 4mk$ (critical damping) $x(t) = (A+Bt)e^{-pt}$ where $p = c/2m$

4) $c^2 < 4mk$ (underdamped) $x(t) = C\cdot\cos(\omega_1 t - \alpha)\cdot e^{-pt}$ where $\omega_1 = \sqrt{\omega_0^2 - p^2}$

Today: $mx'' + cx' + kx = F_0 \cos \omega t$

GOAL: analyse the possible types of solutions as we change ω.

CASE 1: undamped, $\omega \neq \omega_0$

$$m x'' + kx = F_0 \cos \omega t$$

Can use undetermined coefficients to get

$$x(t) = A \cos \omega_0 t + B \sin \omega_0 t + \frac{F_0/m}{\omega_0^2 - \omega^2} \cos \omega t$$

$$= C \cdot \cos(\omega_0 t - \alpha) + \frac{F_0/m}{\omega_0^2 - \omega^2} \cos \omega t$$

What does it look like? Periodic, with higher frequency oscillations within each period.

Special case: BEATS

Happen when $x(0) = x'(0) = 0$

$$x(t) = \frac{F_0}{m(\omega_0^2 - \omega^2)} (\cos \omega t - \cos \omega_0 t)$$

Trick: let's use $\cos(a-b) - \cos(a+b) = 2 \sin a \sin b$, choosing a anb so that

$$a - b = \omega, \; a + b = \omega_0$$

Then $x(t) = \dfrac{F_0}{m(\omega_0^2 - \omega^2)} \sin \dfrac{\omega_0 - \omega}{2} \sin \dfrac{\omega_0 + \omega}{2}$ (look at some graphs!)

CASE 2: undamped, $\omega = \omega_0$ (RESONANCE)

$$mx'' + kx = F_0 \cos \omega_0 t$$

Use undetermined coefficients, guess $x(t) = t(A\cos\omega_0 t + B\sin\omega_0 t)$

$$x(t) = A\cos\omega_0 t + B\sin\omega_0 t + \frac{F_0}{2m\omega_0} t \sin\omega_0 t$$

CASE 3: damped

$$mx'' + cx' + kx = F_0 \cos\omega t$$

$x(t) = x_h(t) + x_p(t)$, where x_h is a solution to $mx'' + cx' + kx = 0$ (transient solution)

x_p is a particular solution (steady solution)

Whatever x_h may be, it's always the case that $\lim_{t\to\infty} x_h(t) = 0$.

x_p has the form $x_p(t) = C\cdot\cos(\omega t - \alpha)$.